On the triangulated category of DQ-modules

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Résumé

Résumé

Cette thèse est consacrée à l'étude des modules de quantification par déformation ou DQ-modules. Elle explore dans quelle mesure certains théorèmes de géométrie algébrique s'étendent aux DQ-modules et plus généralement à un cadre non-commutatif.

Nous établissons un théorème de type Riemann-Roch pour les algèbres différentielles graduées propres et homologiquement lisses, généralisant ainsi un résultat de Shklyarov.

Nous donnons un analogue non-commutatif d'un résultat de Bondal et Van den Bergh affirmant que la catégorie dérivée des faisceaux quasi-cohérents d'une variété algébrique est engendrée par un générateur compact. Il apparaît que la notion d'objet quasi-cohérent n'est pas adaptée à la théorie des DQ-modules. Nous introduisons donc, en nous appuyant sur la notion de complétude cohomologique de Kashiwara-Schapira, la notion d'objet cohomologiquement complet à gradué quasi-cohérent. Nous montrons que ces objets forment une catégorie triangulée, engendrée par un générateur compact et nous en caractérisons les objets compacts.

Nous adaptons au cas des DQ-modules une formule due à Lunts, qui calcule la trace d'un noyau cohérent agissant sur l'homologie de Hochschild d'un DQ-algébroïde. La méthode de Lunts ne semble pas s'appliquer aux DQ-modules. Nous développons donc un formalisme permettant d'obtenir un théorème similaire à celui de Lunts puis nous l'appliquons aux DQ-modules.

Enfin, nous nous intéressons, dans le cadre des DQ-modules, aux transformations intégrales pour lesquelles nous donnons des résultats d'adjonction et démontrons une condition nécessaire et suffisante pour qu'une telle transformation soit une équivalence.

Mots-clefs

Homologie de Hochschild, DQ-algébroïde, DQ-module, catégories triangulée, générateur compact, algèbre différentielle graduée.

On the triangulated category of DQ-modules

Abstract

The main subject of this thesis is the study of deformation quantization modules or DQ-modules. This thesis investigates to which extent some theorems of algebraic geometry can be generalized to DQ-modules. Hence, to a non-commutative setting.

We established a Riemann-Roch type theorem for proper and homologically smooth differential graded algebras which slightly generalizes a result of Shklyarov We give a non-commutative analogue of a result of Bondal and Van den Berg asserting that on a quasi-compact and quasi-separated scheme, the derived category of quasi-coherent sheaves is generated by a single compact generator. It becomes clear that the notion of a quasi-coherent object is not suitable for the theory of DQ-modules. Therefore, relying on the concept of cohomological completenessss of Kashiwara-Schapira, we introduce the notion of cohomologically complete and graded quasi-coherent objects. We show that these objects form a cocomplete triangulated category generated by a single compact generator and we characterize its compact objects.

We adapt to the case of DQ-modules a formula of Lunts which calculates the trace of a coherent kernel acting on the Hochschild homology of a DQ-algebroid stack. The method of Lunts does not seems to work directly in the framework of DQ-modules. We build an abstract formalism in which we obtain a formula similar to Lunts' and we apply this formalism to DQ-modules.

Finally, we study integral transforms in the framework of DQ-modules. In this setting, we recover some adjunction results which are classical in the commutative case. We also give a sufficient and necessary condition for such a transformation to be an equivalence.

Keywords

Hochschild homology, DQ-algebroid, DQ-module, triangulated category, compact generator, differential graded algebra.

Contents

Introduction

Version française

L'objet central de cette thèse est l'étude des modules de quantification par déformation ou DQ-modules. Ils ont été introduits par Kontsevich dans [Kon01] et un travail de fondation a été effectué par Kashiwara et Schapira dans [KS12]. Les DQ-modules sont des modules sur un champ de DQ-algébroïdes. Ces champs sont localement équivalents à des star-algèbres, c'est-à-dire à des quantifications par déformation du faisceau des fonctions holomorphes d'une variété de Poisson complexe.

Cette thèse explore dans quelle mesure certains théorèmes de géométrie algébrique s'étendent aux DQ-modules et donc à un cadre non commutatif. On s'est intéressé en particulier au théorème de Bondal-Van den Bergh concernant les générateurs compacts de la catégorie dérivée des faisceaux quasi-cohérents et à son corollaire affirmant qu'une variété algébrique est dg affine (voir [BvdB03]), à un théorème de Lefschetz dû à V. Lunts (voir [Lun11]) et finalement aux transformations intégrales.

Les notions de nature homologique ou catégorique intervenant en géométrie algébrique peuvent en général être transportées dans un cadre non commutatif. Notre étude s'appuie donc sur l'analyse de certaines structures catégoriques apparaissant dans la théorie des DQ-modules.

Un but, hors de portée à l'heure actuelle, serait d'obtenir un théorème de reconstruction dans l'esprit de celui de Bondal-Orlov pour les variétés symplectiques complexes, à partir de leurs catégories dérivées de DQ-modules cohérents.

La thèse présentée ici comporte quatre chapitres et un appendice traitant des champs en algébroïdes. À l'exception du premier chapitre, ils concernent tous de façon directe la théorie des DQ-modules. Décrivons brièvement leur contenu.

A Riemann-Roch Theorem for dg Algebras

Dans ce chapitre, motivé par la version du théorème de Riemann-Roch pour les algèbres différentielles graduées dû à Shklyarov ([Shk07a]) ainsi que par un résultat similaire obtenu par Kashiwara-Schapira dans le cadre des DQ-modules ([KS12, ch.4, §3]), nous obtenons un théorème de type Riemann-Roch pour les algèbres différentielles graduées faisant intervenir des classes à valeurs dans l'homologie de Hochschild associée à des paires (M, f) où M est un A-module et f un endomorphisme de M.

Le but de ce chapitre, basé sur l'article [Pet10], est d'extraire certains des aspects algébriques de l'approche de Kashiwara-Schapira, afin d'éventuellement fournir un point de vue plus homogène pour attaquer certains théorèmes d'indice.

Nous obtenons une légère généralisation du théorème de Shklyarov. La différence majeure entre notre approche et la sienne est que nous travaillons directement avec l'homologie de Hochschild de l'algèbre différentielle graduée A et non pas avec la définition de l'homologie de Hochschild via la catégorie des complexes parfaits. Finalement, le résultat principal de ce chapitre est

Théorème. *Soit A une algèbre différentielle graduée propre et homologiquement lisse, soit $M \in \mathsf{D}_{\mathrm{perf}}(A)$, $f \in \mathrm{Hom}_A(M, M)$ et $N \in \mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}})$, $g \in \mathrm{Hom}_{A^{\mathrm{op}}}(N, N)$.*

Alors

$$\mathrm{hh}_k(N \overset{\mathrm{L}}{\underset{A}{\otimes}} M, g \overset{\mathrm{L}}{\underset{A}{\otimes}} f) = \mathrm{hh}_{A^{\mathrm{op}}}(N, g) \cup \mathrm{hh}_A(M, f),$$

où $\cup$ est un accouplement entre groupes d'homologies de Hochschild et où $\mathrm{hh}_A(M, f)$ est la classe de Hochschild de la paire (M, f) à valeurs dans l'homologie de Hochschild de A.

DG Affinity of DQ-modules

Dans ce chapitre qui est extrait de l'article [Pet11a], nous donnons un analogue non-commutatif à un célèbre résultat de Bondal et Van den Bergh affirmant que, sur un schéma quasi-séparé et quasi-compact, la catégorie $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ est engendrée par un générateur compact et que les objets compacts de $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ sont les complexes parfaits. Un corollaire de ce résultat est que la catégorie $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ est équivalente à la catégorie dérivée d'une algèbre différentielle graduée convenable.

Il apparaît assez rapidement que la notion d'objet quasi-cohérent n'est pas adaptée à la théorie des DQ-modules algébriques. Une première difficulté consiste donc à trouver une catégorie triangulée qui puisse remplace la catégorie $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$. Un ingrédient essentiel est la notion de complétude cohomologique due à Kashiwara-Schapira. Nous introduisons donc la notion d'objet cohomologiquement complet à gradué quasi-cohérent. Ces objets sont appelés qcc modules et forment une catégorie triangulée $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ (où $\mathcal{A}_X$ est un DQ-algébroïde) qui peut être considérée comme la déformation de $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ lorsque l'on déforme $\mathcal{O}_X$ en $\mathcal{A}_X$. Une seconde difficulté réside dans la cocomplétude de la catégorie des qcc. En effet $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ est cocomplète non pas pour la somme directe usuelle de la catégorie des DQ-modules mais pour la complétion cohomologique de cette somme directe.

Deux foncteurs jouent un rôle fondamental dans la théorie des DQ-modules. Les foncteurs

$$\begin{aligned} \iota_g : \mathsf{D}(\mathcal{O}_X) &\to \mathsf{D}(\mathcal{A}_X) \\ \mathcal{M} &\mapsto {}_A\mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M} \end{aligned} \qquad\qquad \begin{aligned} \mathrm{gr}_\hbar : \mathsf{D}(\mathcal{A}_X) &\to \mathsf{D}(\mathcal{O}_X) \\ \mathcal{M} &\mapsto \mathcal{O}_{XA} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}. \end{aligned}$$

L'observation essentielle est qu'un générateur compact de la catégorie $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ va fournir un générateur compact de la catégorie des qcc. En effet, on a

Proposition. *Si $\mathcal{G}$ est un générateur compact de $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ alors $\iota_g(\mathcal{G})$ est un générateur compact de $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.*

L'existence d'un générateur compact de $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ étant garantie par [BvdB03]. En nous appuyant sur un résultat de Ravenel et Neeman ([Rav84] et [Nee92]), nous caractérisons ensuite les objets compacts de $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.

Théorème. *Un object $\mathcal{M}$ de $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ est compact si et seulement si $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ et $\mathcal{A}_X^{loc} \otimes_{\mathcal{A}_X} \mathcal{M} = 0$.*

Finalement, en utilisant un résultat dû à Keller [Kel94], on déduit de tout ceci

Théorème. *Soit X une variété complexe lisse, munie d'un DQ-algebroïde $\mathcal{A}_X$. Alors $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ est équivalente à $\mathsf{D}(\Lambda)$ où Λ est une algèbre différentielle graduée convenable dont la cohomologie est bornée.*

Enfin, nous étudions la catégorie des qcc lorsque l'on suppose la variété affine. On montre alors que l'équivalence de catégories triangulées entre $\mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X)$ et $\mathsf{D}^+(\mathcal{O}_X(X))$ se relève en une équivalence entre $\mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X)$ et la catégorie triangulée $\mathsf{D}^+_{\mathrm{cc}}(\mathcal{A}_X(X))$ des $\mathcal{A}_X(X)$-modules cohomologiquement complets.

Théorème. *Soit X une variété algébrique affine lisse. On pose $A = \Gamma(X, \mathcal{A}_X)$ et $B = \Gamma(X, \mathcal{O}_X)$.*

Les foncteurs

$$\Phi : \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathsf{D}^+_{\mathrm{cc}}(A),\ \Phi(\mathcal{M}) = \mathrm{R}\Gamma(X, \mathcal{M})$$

et

$$\Psi : \mathsf{D}^+_{\mathrm{cc}}(A) \to \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X), \Psi(M) = (\mathcal{A}_X \otimes_{A_X} a_X^{-1} M)^{\mathrm{cc}}$$

sont des équivalences de catégories triangulées et sont inverses l'un de l'autre. De plus le diagramme ci-dessous est quasi-commutatif.

$$\begin{array}{ccc}
\mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X) & \overset{\Phi}{\underset{\Psi}{\rightleftarrows}} & \mathsf{D}^+_{\mathrm{cc}}(A) \\
\downarrow{\scriptstyle \mathrm{gr}_\hbar} & & \downarrow{\scriptstyle \mathrm{gr}_\hbar} \\
\mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X) & \overset{\mathrm{R}\Gamma(X,\cdot)}{\underset{\mathcal{O}_X \otimes_{B_X} \cdot}{\rightleftarrows}} & \mathsf{D}^+(B).
\end{array}$$

The Lefschetz-Lunts formula for DQ-modules

Dans ce chapitre, extrait de [Pet11b], on adapte aux cas des DQ-modules une formule, due à V. Lunts, qui calcule la trace d'un noyau cohérent agissant sur l'homologie de Hochschild. La méthode de V. Lunts ne semble pas s'appliquer directement au cas des DQ-modules. En effet, dans le cadre commutatif, si X est une variété projective lisse, le morphisme $X \to \mathrm{pt}$ va permettre d'intégrer une classe. Cependant, dans le cadre des DQ-modules un tel morphisme n'existe pas et l'on doit intégrer une paire de classes.

Pour remédier à ce problème, on s'inspire de l'idée courante en topologie algébrique (cf. [LMSM86]) selon laquelle les théorèmes du point fixe de Lefschetz sont liés à des structures monoïdales. Dans la première partie du chapitre, on développe un formalisme abstrait permettant d'obtenir un théorème similaire à celui de Lunts à partir d'une catégorie monoïdale munie d'un foncteur à valeurs dans la catégorie dérivée des k-modules et satisfaisant un certain nombre de propriétés (voir la sous-section 3.2.2). On applique ensuite ce formalisme aux DQ-modules et on obtient

Théorème. *Soit X une variété analytique complexe lisse et compact munie d'un DQ-algébroïde $\mathcal{A}_X$. Soit $\lambda \in \mathrm{HH}_0(\mathcal{A}_{X \times X^a})$. Considérons le morphisme (3.3.14)*

$$\Phi_\lambda : \mathbb{HH}(\mathcal{A}_X) \to \mathbb{HH}(\mathcal{A}_X).$$

alors

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\lambda) = \mathrm{hh}_{X^a \times X}(\mathcal{C}_{X^a}) \underset{X \times X^a}{\cup} \lambda.$$

Proposition. *Soit X une variété analytique complexe lisse et compacte munie d'un DQ-algébroïde $\mathcal{A}_X$ et $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X\times X^a})$. Alors*

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_{\mathcal{K}}) = \chi(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X\times X^a}}{\otimes}} \mathcal{K})).$$

On donne ensuite différentes formes de ces formules en particulier lorsque la variété est supposée symplectique. On retrouve aussi les résultats de Lunts.

Fourier-Mukai transforms for DQ-modules

Dans ce dernier chapitre, qui provient d'un travail en cours, on s'intéresse aux transformations intégrales pour les DQ-modules. Ce travail est motivé par le théorème de reconstruction de Bondal-Orlov. On commence par retrouver, avec une méthode différente de la méthode usuelle, certains résultats d'adjonction pour les transformations intégrales.

Proposition. *Soit X_1 (resp. X_2) une variété complexe projective lisse équipée de la topologie de Zariski et munie d'un DQ-algébroïde $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Soit $\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ la transformation intégrale associée à $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$ et $\Phi_{\mathcal{K}_R} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$ (resp. $\Phi_{\mathcal{K}_L} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$) la transformation intégrale associée à $\mathcal{K}_R = \mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{2^a}{\circ} \omega_{2^a}$ (resp. $\mathcal{K}_L = \mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{1}{\circ} \omega_1$). Alors $\Phi_{\mathcal{K}_R}$ (resp. $\Phi_{\mathcal{K}_L}$) est adjoint à droite (resp. à gauche) de $\Phi_{\mathcal{K}}$.*

Le résultat central du chapitre est une condition pour qu'une transformation intégrale soit une équivalence de catégories.

Théorème. *Soit X_1 (resp. X_2) une variété complexe projective lisse équipée de la topologie de Zariski et munie d'un DQ-algébroïde $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Soit $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$. Les conditions suivantes sont équivalentes.*

(i) Le foncteur $\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ est pleinement fidèle (resp. est une équivalence de catégories triangulées).

(ii) Le foncteur $\Phi_{\mathrm{gr}_\hbar \mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_1)$ est pleinement fidèle (resp. est une équivalence de catégories triangulées).

La preuve de l'implication $(ii) \Rightarrow (i)$ ne présente pas de difficulté et fait appel à des arguments classiques de théorie des DQ-modules. Quant à la preuve de l'implication $(i) \Rightarrow (ii)$, elle repose essentiellement sur l'observation suivante.

Proposition. *Soit X une variété algébrique complexe lisse munie d'un DQ-algebroïde.*

(i) Si $\mathcal{G}$ est un générateur compact de $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ alors $\mathrm{gr}_\hbar \iota_g \mathcal{G}$ est encore un générateur compact de $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$.

(ii) On a $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X) = \langle \mathrm{gr}_\hbar \iota_g(\mathcal{G}) \rangle$.

En effet, la preuve de l'implication $(i) \Rightarrow (ii)$ revient alors à considérer la sous-catégorie pleine de $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$ dont les objets sont ceux pour lesquels $\Phi_{\mathrm{gr}_\hbar \mathcal{K}}$ est pleinement fidèle. On montre que cette catégorie est une sous-catégorie épaisse de $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$ contenant $\mathrm{gr}_\hbar \iota_g(\mathcal{G})$. La proposition ci-dessus permet alors de conclure.

English version

The main subject of this thesis is the study of deformation quantization modules (DQ-modules for short). They have been introduced by Konstevich in [Kon01] and a important foundational work has been carried out by Kashiwara and Schapira in [KS12]. DQ-modules are modules over a DQ-algebroid stacks. These stacks are locally equivalent to star-algebras that is to say to deformation quantization of the sheaf of holomorphic functions of a complex Poisson variety.

This thesis investigates in which extents some theorems of algebraic geometry can be generalised to DQ-modules. Hence, to a non-commutative setting. We have focus our attention on a result of Bondal-Van den Bergh concerning the compact generators of the derived category of quasi-coherent sheaves and on its corollary asserting that an algebraic variety is dg affine (see [BvdB03]), on a Lefschetz theorem due to V. Lunts (see [Lun11]) and finally on integral transforms.

Notions of homological or categorical nature, appearing in algebraic geometry, may usually be transposed to non-commutative framework. Hence, our study relies on an analysis of certain categorical structures showing up in the theory of DQ-modules.

A goal, out of reach at the present time, would be to obtain a reconstruction theorem for complex symplectic varieties starting from their derived category of coherent DQ-modules. Such a result is motivated by Bondal-Orlov's famous reconstruction result.

The present thesis is made of four chapters and one appendix concerning algebroid stacks. With the exception of the first chapter, they all directly concern the theory of DQ-modules. Let us briefly describe them.

A Riemann-Roch Theorem for dg Algebras

In this chapter, motivated by the Riemann-Roch theorem for differential graded algebras due to Shklyarov [Shk07a] and by a similar result of Kashiwara-Schapira for DQ-modules ([KS12, §4.3]), we obtain a Riemann-Roch theorem for differential graded algebras involving classes of pairs (M, f) where M is a perfect A-module and f an endomorphism of M. These classes are with values in the Hochschild homology of the algebra A.

The aim of this chapter, based on [Pet10], is to extract some of the algebraic aspects of this latter approach with the hope that the resulting techniques will provide a uniform point of view for proving some index theorems.

We obtain a slightly more general version of Shklyarov's theorem. The major difference between his approach and ours is that we are directly working with the Hochschild homology of the differential graded algebra A and not with the definition of Hochschild homology in terms of the category of perfect complexes. Finally, the main result of this chapter is

Theorem. *Let A be a proper, homologically smooth dg algebra, $M \in \mathsf{D}_{\mathrm{perf}}(A)$, $f \in \mathrm{Hom}_A(M,M)$ and $N \in \mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}})$, $g \in \mathrm{Hom}_{A^{\mathrm{op}}}(N,N)$.*

Then

$$\mathrm{hh}_k(N \overset{\mathrm{L}}{\underset{A}{\otimes}} M, g \overset{\mathrm{L}}{\underset{A}{\otimes}} f) = \mathrm{hh}_{A^{\mathrm{op}}}(N,g) \cup \mathrm{hh}_A(M,f),$$

where $\cup$ is a pairing between the corresponding Hochschild homology groups and where $\mathrm{hh}_A(M,f)$ is the Hochschild class of the pair (M,f) with value in the Hochschild homology of A.

DG Affinity of DQ-modules

In this chapter, extracted from the [Pet11a], we give a non-commutative analogue of a famous result of Bondal-Van den Berg asserting that, on a quasi-compact and quasi-separated scheme, the category $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is generated by a compact generator and that the compact object of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ are the perfect complexes. This result implies that the category $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is equivalent to the derived category of a suitable differential graded algebra.

It becomes rapidly clear that the notion of quasi-coherent object is not suitable for the theory of DQ-modules. A first difficulty is to find a triangulated category to replace $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$. A key ingredient is Kashiwara-Schapira's notion of cohomological completeness. Therefore, we introduce the notion of cohomologically complete and graded quasi-coherent objects. These objects are called qcc modules and form a triangulated category $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ (where $\mathcal{A}_X$ is a DQ-algebroid) which can be thought of as the deformation of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ while deforming $\mathcal{O}_X$ into $\mathcal{A}_X$. A second difficulty lies in the cocompletness of the category of qcc objects. Indeed, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is not a cocomplete category for the usual direct sum of the category of DQ-modules but for the cohomological completion of this direct sum.

Two functors play a leading role in the theory of DQ-modules. The functors

$$\begin{aligned} \iota_g : \mathsf{D}(\mathcal{O}_X) &\to \mathsf{D}(\mathcal{A}_X) & \qquad \mathrm{gr}_\hbar : \mathsf{D}(\mathcal{A}_X) &\to \mathsf{D}(\mathcal{O}_X) \\ \mathcal{M} &\mapsto {}_A\mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M} & \mathcal{M} &\mapsto \mathcal{O}_{XA} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}. \end{aligned}$$

A key observation is that a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ gives a compact generator of the qcc objects. Indeed, we have

Proposition. *If $\mathcal{G}$ is a generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$, then $\iota_g(\mathcal{G})$ is a generator of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.*

The existence of a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is granted by [BvdB03]. Relying on a theorem of Ravenel and Neeman (see [Rav84] and [Nee92]) we describe completely the compact objects of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.

Theorem. *An object $\mathcal{M}$ of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is compact if and only if $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ and $\mathcal{A}_X^{loc} \otimes_{\mathcal{A}_X} \mathcal{M} = 0$.*

Finally, using a result of Keller [Kel94], we deduce that

Theorem. *Assume X is a smooth complex algebraic variety endowed with a deformation algebroid $\mathcal{A}_X$. Then, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is equivalent to $\mathsf{D}(\Lambda)$ for a suitable dg algebra Λ with bounded cohomology.*

Finally, we study qcc sheaves on an affine variety and prove that the equivalence of triangulated categories between $\mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X)$ and $\mathsf{D}^+(\mathcal{O}_X(X))$ lifts to an equivalence between $\mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X)$ and the triangulated category $\mathsf{D}^+_{cc}(\mathcal{A}_X(X))$ of cohomologically complete $\mathcal{A}_X(X)$-modules

Theorem. *Let X be a smooth affine variety. We set $A = \Gamma(X, \mathcal{A}_X)$ and $B = \Gamma(X, \mathcal{O}_X)$. The functors*

$$\Phi : \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathsf{D}^+_{\mathrm{cc}}(A),\ \Phi(\mathcal{M}) = \mathrm{R}\Gamma(X, \mathcal{M})$$

and

$$\Psi : \mathsf{D}^+_{\mathrm{cc}}(A) \to \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X), \Psi(M) = (\mathcal{A}_X \otimes_{A_X} a_X^{-1} M)^{\mathrm{cc}}$$

are equivalences of triangulated categories, are inverses one to each other and the diagram below is quasi-commutative

$$\begin{tikzcd} \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X) \arrow[r, shift left, "\Phi"] \arrow[d, "\mathrm{gr}_\hbar"] & \mathsf{D}^+_{\mathrm{cc}}(A) \arrow[l, shift left, "\Psi"] \arrow[d, "\mathrm{gr}_\hbar"] \\ \mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X) \arrow[r, shift left, "\mathrm{R}\Gamma(X,\cdot)"] & \mathsf{D}^+(B). \arrow[l, shift left, "\mathcal{O}_X \otimes_{B_X} \cdot"] \end{tikzcd}$$

The Lefschetz-Lunts formula for DQ-modules

In this chapter, extracted from [Pet11b], we adapt to the case of DQ-modules a formula of V. Lunts which calculates the trace of a coherent kernel acting on the Hochschild homology. The method of V. Lunts does not seems to work directly in the framework of DQ-modules. Indeed, in the commutative case, if X is a smooth projective variety, the morphism $X \to \mathrm{pt}$ allows to integrate a class. However, such a map does not exist in the theory of DQ-modules. Thus, it is not possible to integrate a single class with value in Hochschild homology and one has to integrate a pair of classes.

To solve this problem, we follow the idea from algebraic topology (cf. [LMSM86]) according to which Lefschetz fixed point theorems are linked to some monoidal structures. In the first part of the chapter, we build an abstract formalism in which we can get a formula for the trace of a class acting on a certain homology, starting from a symmetric monoidal category endowed with some specific data (see subsection 3.2.2). Then, we apply this formalism to DQ-modules and get

Theorem. *Let X be a complex compact manifold endowed with a DQ-algebroid $\mathcal{A}_X$. Let $\lambda \in \mathrm{HH}_0(\mathcal{A}_{X\times X^a})$. Consider the map (3.3.14)*

$$\Phi_\lambda : \mathbb{HH}(\mathcal{A}_X) \to \mathbb{HH}(\mathcal{A}_X).$$

Then

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\lambda) = \mathrm{hh}_{X^a\times X}(\mathcal{C}_{X^a}) \underset{X\times X^a}{\cup} \lambda.$$

Proposition. *Let X be a complex compact manifold endowed with a DQ-algebroid $\mathcal{A}_X$ and let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X\times X^a})$. Then*

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\mathcal{K}) = \chi(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X\times X^a}}{\otimes}} \mathcal{K})).$$

Finally, we give different forms of these formulas, especially when the manifolds is assumed to be symplectic. We also recover Lunts'results.

Fourier-Mukai transforms for DQ-modules

In this last chapter, that comes from an ongoing work, we focus our attention on integral transforms for DQ-modules. This work is motivated by Bondal-Orlov's reconstruction theorem. We start by recovering some adjunction results for integral transforms.

Proposition. *Let X_1 (resp. X_2) be a smooth complex projective variety endowed with the Zariski topology and equipped with a DQ-algebroid $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Let $\Phi_\mathcal{K} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ be the Fourier-Mukai functor associated to $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$ and $\Phi_{\mathcal{K}_R} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$ (resp. $\Phi_{\mathcal{K}_L} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$) the Fourier-Mukai functor associated to $\mathcal{K}_R = \mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{2^a}{\circ} \omega_{2^a}$ (resp. $\mathcal{K}_L = \mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{1}{\circ} \omega_1$). Then $\Phi_{\mathcal{K}_R}$ (resp. $\Phi_{\mathcal{K}_L}$) is right (resp. left) adjoint to $\Phi_\mathcal{K}$.*

The main result of the chapter is a condition for an integral transform to be an equivalence of category.

Theorem. *Let X_1 (resp. X_2) be a smooth complex projective variety endowed with the Zariski topology and equipped with a DQ-algebroid $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$. The following conditions are equivalent*

(i) The functor $\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ is fully faithful (resp. an equivalence of triangulated categories).

(ii) The functor $\Phi_{\mathrm{gr}_\hbar \mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_1)$ is fully faithful (resp. an equivalence of triangulated categories).

The proof of the implication $(ii) \Rightarrow (i)$ does not present any difficulties and use classical techniques of DQ-modules theory. The proof of the implication $(i) \Rightarrow (ii)$ relies essentially on the following observation.

Proposition. *Let X be a smooth complex algebraic variety endowed with a DQ-algebroid.*

(i) If $\mathcal{G}$ is a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ then $\mathrm{gr}_\hbar \iota_g \mathcal{G}$ is still a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$.

(ii) One has $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X) = \langle \mathrm{gr}_\hbar \iota_g(\mathcal{G}) \rangle$.

The proof of the implication $(i) \Rightarrow (ii)$ comes down to study the full subcategory of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$ whose objects are those for which $\Phi_{\mathrm{gr}_\hbar \mathcal{K}}$ is fully faithful. We show that this category is a thick subcategory of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$ containing $\mathrm{gr}_\hbar \iota_g(\mathcal{G})$. The above proposition leads to the result.

Chapter 1

A Riemann-Roch Theorem for dg Algebras

1.1 Introduction

An algebraic version of the Riemann-Roch formula was recently obtained by D. Shklyarov [Shk07a] in the framework of the so-called noncommutative derived algebraic geometry. More precisely, motivated by the well known result of A. Bondal and M. Van den Bergh about "dg-affinity" of classical varieties, D. Shklyarov has obtained a formula for the Euler characteristic of the Hom-complex between two perfect modules over a dg-algebra in terms of the Euler classes of the modules.

On the other hand, M. Kashiwara and P. Schapira [KS12] initiated an approach to the Riemann-Roch theorem in the framework of deformation quantization modules (DQ-modules) with the view towards applications to various index type theorems. Their approach is based on Hochschild homology which, in this setup, admits a description in terms of the dualizing complexes in the derived categories of coherent DQ-modules.

The present chapter, based on the paper [Pet10], is an attempt to extract some algebraic aspects of this latter approach with the hope that the resulting techniques will provide a uniform point of view for proving Riemann-Roch type results for DQ-modules, D-modules etc. (e.g. the Riemann-Roch-Hirzebruch formula for traces of differential operators obtained by M. Engeli and G. Felder [EF08]). In this chapter, we obtain a Riemann-Roch theorem in the dg setting, similarly as D. Shklyarov. However, our approach is really different of the latter one in that we avoid the categorical definition of the Hochschild homology, and use instead the Hochschild homology of the ring A expressed in terms of dualizing objects. Our result is slightly more general than the one obtained in [Shk07a]. Instead of a kind of non-commutative Riemann-Roch theorem, we rather prove a kind of non-commutative Lefschetz theorem. Indeed, it involves certain Hochschild classes of *pairs* (M, f) where M is a perfect dg module over a smooth proper dg algebra and f is an endomorphism of M in the derived category of perfect A-modules. Moreover, our approach follows [KS12]. In particular, we have in our setting relative finiteness and duality results (Theorem 1.3.16 and Theorem 1.3.27) that may be compared with [KS12, Theorem 3.2.1] and [KS12, Theorem 3.3.3]. Notice that the idea to approach the classical Riemann-Roch theorem for smooth projective varieties via their Hochschild homology goes back at least to the work of N. Markarian [Mar06]. This approach was developed further by A. Căldăraru [Căl03], [Căl05] and A. Căldăraru, S. Willerton [CW10] where, in particular, certain purely categorical aspects of the story were emphasized. The results of [Căl03] suggested that a Riemann-Roch type formula might exist for triangulated categories of quite general nature, provided they possess Serre duality. In this categorical framework, the role of the Hochschild homology is played by the space of morphisms from the inverse of the Serre functor to the identity endofunctor. In a sense, our result can be viewed as a non-commutative generalization of A. Căldăraru's version of the topological Cardy condition [Căl03]. Our original motivation was different though it also came from the theory of DQ-modules [KS12].

Here is our main result:

Theorem. *Let A be a proper, homologically smooth dg algebra, $M \in \mathsf{D}_{\mathrm{perf}}(A)$, $f \in \mathrm{Hom}_A(M,M)$ and $N \in \mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}})$, $g \in \mathrm{Hom}_{A^{\mathrm{op}}}(N,N)$.*
Then

$$\mathrm{hh}_k(N \overset{\mathrm{L}}{\underset{A}{\otimes}} M, g \overset{\mathrm{L}}{\underset{A}{\otimes}} f) = \mathrm{hh}_{A^{\mathrm{op}}}(N,g) \cup \mathrm{hh}_A(M,f),$$

where $\cup$ is a pairing between the corresponding Hochschild homology groups and where $\mathrm{hh}_A(M,f)$ is the Hochschild class of the pair (M,f) with value in the Hochschild homology of A.

The above pairing is obtained using Serre duality in the derived category of perfect complexes and, thus, it strongly resembles analogous pairings, studied in some of the references previously mentioned (cf. [Căl03], [KS12], [Shk07b]). We prove that various methods of constructing a pairing on Hochschild homology lead to the same result. Notice that in [Ram10], A. Ramadoss studied the links between different pairing on Hochschild homology.

To conclude, we would like to mention the recent paper by A. Polishchuk and A. Vaintrob [PV10] where a categorical version of the Riemann-Roch theorem was applied in the setting of the so-called Landau-Ginzburg models (the categories of matrix factorizations). We hope that our results, in combination with some results by D. Murfet [Mur09], may provide an alternative way to derive the Riemann-Roch formula for singularities.

1.2 Conventions

All along this chapter k is a field of characteristic zero. A k-algebra is a k-module A equiped with an associative k-linear multiplication admitting a two sided unit 1_A.

All the graded modules considered in this chapter are cohomologically $\mathbb{Z}$-graded. We abreviate differential graded module (resp. algebra) by dg module (resp. dg algebra).

If A is a dg algebra and M and N are dg A-modules, we write $\mathrm{Hom}_A^{\bullet}(M,N)$ for the total Hom-complex.

If M is a dg k-module, we define $M^* = \mathrm{Hom}_k^{\bullet}(M,k)$ where k is considered as the dg k-module whose 0-th components is k and other components are zero.

We write $\otimes$ for the tensor product over k. If x is an homogeneous element of a differential graded module we denote by $|x|$ its degree.

If A is a dg algebra we will denote by A^{op} the opposite dg algebra. It is the same as a differential graded k-module but the multiplication is given by $a \cdot b = (-1)^{|a||b|} ba$. We denote by A^e the dg algebra $A \otimes A^{\mathrm{op}}$ and by ${}^eA$ the algebra $A^{\mathrm{op}} \otimes A$.

By a module we understand a left module unless otherwise specified . If A and B are dg algebras, A-B bimodules will be considered as left $A \otimes B^{\mathrm{op}}$-modules via the action

$$a \otimes b \cdot m = (-1)^{|b||m|} amb.$$

If we want to emphasize the left (resp. right) structure of an A-B bimodule we will write $_A M$ (resp. M_B). If M is an $A \otimes B^{\mathrm{op}}$-modules, then we write M^{op} for the corresponding $B^{\mathrm{op}} \otimes A$-module. Notice that $(M^{\mathrm{op}})^* \simeq (M^*)^{\mathrm{op}}$ as $B \otimes A^{\mathrm{op}}$-modules.

1.3 Perfect modules

1.3.1 Compact objects

We recall some classical facts concerning compact objects in triangulated categories. We refer the reader to [Nee01].

Let $\mathcal{T}$ be a triangulated category admitting arbitrary small coproducts.

Definition 1.3.1. An object M of $\mathcal{T}$ is compact if for each family $(M_i)_{i \in I}$ of objects of $\mathcal{T}$ the canonical morphism

$$\bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{T}}(M, M_i) \to \mathrm{Hom}_{\mathcal{T}}(M, \bigoplus_{i \in I} M_i) \tag{1.3.1}$$

is an isomorphism. We denote by $\mathcal{T}^c$ the full subcategory of $\mathcal{T}$ whose objects are the compact objects of $\mathcal{T}$.

Recall that a triangulated subcategory of $\mathcal{T}$ is called thick if it is closed under isomorphisms and direct summands.

Proposition 1.3.2. *The category $\mathcal{T}^c$ is a thick subcategory of $\mathcal{T}$.*

We prove the following fact that will be of constant use.

Proposition 1.3.3. *Let $\mathcal{T}$ and $\mathcal{S}$ be two triangulated categories and F and G two functors of triangulated categories from $\mathcal{T}$ to $\mathcal{S}$ with a natural transformation $\alpha : F \Rightarrow G$. Then the full subcategory $\mathcal{T}_\alpha$ of $\mathcal{T}$ whose objects are the X such that $\alpha_X : F(X) \to G(X)$ is an isomorphism is a thick subcategory of $\mathcal{T}$.*

Proof. The category $\mathcal{T}_\alpha$ is non-empty since 0 belongs to it and it is stable by shift since F and G are functors of triangulated categories. Moreover, the category $\mathcal{T}_\alpha$ is a full subcategory of a triangulated category. Thus, to verify that $\mathcal{T}_\alpha$ is triangulated, it only remains to check that it is stable by taking cones. Let $f : X \to Y$ be a morphism of $\mathcal{T}_\alpha$. Consider a distinguished triangle in $\mathcal{T}_\alpha$

$$X \xrightarrow{f} Y \to Z \to X[1].$$

We have the following diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X", "\wr"'] & F(Y) \arrow[r] \arrow[d, "\alpha_Y", "\wr"'] & F(Z) \arrow[r] \arrow[d, "\alpha_Z"] & F(X[1]) \arrow[d, "\alpha_{X[1]}", "\wr"'] \\
G(X) \arrow[r, "G(f)"] & G(Y) \arrow[r] & G(Z) \arrow[r] & G(X[1]).
\end{tikzcd}$$

By the five Lemma it follows that the morphism α_Z is an isomorphism. Therefore, Z belongs to $\mathcal{T}_\alpha$. This implies that the category $\mathcal{T}_\alpha$ is triangulated.

It is clear that $\mathcal{T}_\alpha$ is closed under isomorphism. Since F and G are functors of triangulated categories they commutes with finite direct sums. Thus, $\mathcal{T}_\alpha$ is stable under taking direct summands. It follows that $\mathcal{T}_\alpha$ is a thick subcategory of $\mathcal{T}$. □

Definition 1.3.4. The triangulated category $\mathcal{T}$ is compactly generated if there is a set $\mathcal{G}$ of compact objects G such that an object M of $\mathcal{T}$ vanishes if and only if we have $\mathrm{Hom}_{\mathcal{T}}(G[n], M) \simeq 0$ for every $G \in \mathcal{G}$ and $n \in \mathbb{Z}$.

Theorem 1.3.5 ([Nee92, Rav84])**.** *Let $\mathcal{G}$ be as in Definition 1.3.4. An object of $\mathcal{T}$ is compact if and only if it is isomorphic to a direct factor of an iterated triangle extension of copies of object of $\mathcal{G}$ shifted in both directions.*

Remark 1.3.6. The above theorem implies that the category of compact objects is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{G}$.

1.3.2 The category of perfect modules

In this section, following [Kel06], we recall the definition of the category of perfect modules.

Let A be a differential graded algebra. One associates to A its category of differential graded modules, denoted $\mathcal{C}(A)$, whose objects are the differential graded modules and whose morphisms are the morphisms of chain complexes.

Recall that the category $\mathcal{C}(A)$ has a compactly generated model structure, called the projective structure, where the weak equivalences are the quasi-isomorphisms, the fibrations are the level-wise surjections. The reader may refer to [Hov99] for model categories and to [Fre09, ch.11] for a detailed account on the projective model stucture of $\mathcal{C}(A)$.

The derived category $\mathsf{D}(A)$ is the localisation of $\mathcal{C}(A)$ with respects to the class of quasi-isomorphisms. The category $\mathsf{D}(A)$ is a triangulated category, it admits arbitrary coproducts and is compactly generated by the object A. Theorem 1.3.5 leads to Proposition 1.3.7 which allows us to define perfect modules in terms of compact objects.

Proposition 1.3.7. *An object of $\mathsf{D}(A)$ is compact if and only if it is isomorphic to a direct factor of an iterated extension of copies of A shifted in both directions.*

Definition 1.3.8. A differential graded module is perfect if it is a compact object of $\mathsf{D}(A)$. We write $\mathsf{D}_{\mathrm{perf}}(A)$ for the category of compact objects of $\mathsf{D}(A)$.

Remark 1.3.9. This definition implies immediatly that M is a perfect k-module if and only if $\sum_i \dim_k \mathrm{H}^i(M) < \infty$.

A direct consequence of Proposition 1.3.7 is

Proposition 1.3.10. *Let A and B be two dg algebras and $F : \mathsf{D}(A) \to \mathsf{D}(B)$ a functor of triangulated categories. Assume that $F(A)$ belongs to $\mathsf{D}_{\mathrm{perf}}(B)$. Then, for any X in $\mathsf{D}_{\mathrm{perf}}(A)$, $F(X)$ is an object of $\mathsf{D}_{\mathrm{perf}}(B)$.*

Proposition 1.3.11. *Let A and B be two dg algebras and $F, G : \mathsf{D}(A) \to \mathsf{D}(B)$ two functors of triangulated categories and $\alpha : F \Rightarrow G$ a natural transformation. If $\alpha_A : F(A) \to G(A)$ is an isomorphism then $\alpha_M : F(M) \to G(M)$ is an isomorphism for every $M \in \mathsf{D}_{\mathrm{perf}}(A)$.*

Proof. By Proposition 1.3.3, the category $\mathcal{T}_\alpha$ is thick. This category contains A by hypothesis. It follows by Remark 1.3.6 that $\mathcal{T}_\alpha = \mathsf{D}_{\mathrm{perf}}(A)$. □

Definition 1.3.12. A dg A-module M is a finitely generated semi-free module if it can be obtain by iterated extensions of copies of A shifted in both directions.

Proposition 1.3.13. *(i) Finitely generated semi-free modules are cofibrant objects of $\mathcal{C}(A)$ endowed with the projective structure.*

(ii) A perfect module is quasi-isomorphic to a direct factor of a finitely generated semi-free module.

Proof. (i) is a direct consequence of [Fre09, Proposition 11.2.9].

(ii) follows from Proposition 1.3.7 and from the facts that, in the projective structure, every object is fibrant and finitely generated semi-free modules are cofibrant.

□

Remark 1.3.14. The above statement is a special case of [TV07, Proposition 2.2].

1.3.3 Finiteness results for perfect modules

We summarize some finiteness results for perfect modules over a dg algebra satisfying suitable finiteness and regularity hypothesis. The main reference for this section is [TV07]. Most of the statements of this subsection and their proofs can be found in greater generality in [TV07, §2.2]. For the sake of completeness, we give the proof of these results in our specific framework.

Definition 1.3.15. A dg k-algebra A is said to be proper if it is perfect over k.

The next theorem, though the proof is much easier, can be thought as a dg analog to the theorem asserting the finiteness of proper direct images for coherent $\mathcal{O}_X$-modules.

Theorem 1.3.16. *Let A, B and C be dg algebras. Assume B is a proper dg algebra. Then the functor $\cdot \overset{\mathrm{L}}{\underset{B}{\otimes}} \cdot : \mathsf{D}(A \otimes B^{\mathrm{op}}) \times \mathsf{D}(B \otimes C^{\mathrm{op}}) \to \mathsf{D}(A \otimes C^{\mathrm{op}})$ induces a functor $\cdot \overset{\mathrm{L}}{\underset{B}{\otimes}} \cdot : \mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}}) \times \mathsf{D}_{\mathrm{perf}}(B \otimes C^{\mathrm{op}}) \to \mathsf{D}_{\mathrm{perf}}(A \otimes C^{\mathrm{op}})$.*

Proof. According to Proposition 1.3.10, we only need to check that $(A \otimes B^{\mathrm{op}}) \overset{\mathrm{L}}{\underset{B}{\otimes}} (B \otimes C^{\mathrm{op}}) \simeq A \otimes B^{\mathrm{op}} \otimes C^{\mathrm{op}} \in \mathsf{D}_{\mathrm{perf}}(A \otimes C^{\mathrm{op}})$. In $\mathcal{C}(k)$, B is homotopically equivalent to $\mathrm{H}(B) := \bigoplus_{n \in \mathbb{Z}} \mathrm{H}^n(B)[n]$ since k is a field. Then, in $\mathcal{C}(A \otimes C^{\mathrm{op}})$, $A \otimes B^{\mathrm{op}} \otimes C^{\mathrm{op}}$ is homotopically equivalent to $A \otimes \mathrm{H}(B^{\mathrm{op}}) \otimes C^{\mathrm{op}}$ which is a finitely generated free $A \otimes C^{\mathrm{op}}$-module since B is proper. □

We recall a regularity condition for dg algebra called homological smoothness, [KS09], [TV07].

Definition 1.3.17. A dg-algebra A is said to be homologically smooth if $A \in \mathsf{D}_{\mathrm{perf}}(A^e)$.

Proposition 1.3.18. *The tensor product of two homologically smooth dg-algebras is an homologically smooth dg-algebra.*

Proof. Obvious. □

There is the following characterization of perfect modules over a proper homologically smooth dg algebra extracted from [TV07, Corollary 2.9].

Theorem 1.3.19. *Let A be a proper dg algebra. Let $N \in \mathsf{D}(A)$.*

(i) If $N \in \mathsf{D}_{\mathrm{perf}}(A)$ then N is perfect over k.

(ii) If A is homologically smooth and N is perfect over k then $N \in \mathsf{D}_{\mathrm{perf}}(A)$.

Proof. We follow the proof of [Shk07b].

(i) Apply Proposition 1.3.10.

(ii) Assume that $N \in \mathsf{D}(A)$ is perfect over k. Let us show that the triangulated functor $\cdot \overset{\mathrm{L}}{\underset{A}{\otimes}} N : \mathsf{D}(A^e) \to \mathsf{D}(A)$ induces a triangulated functor $\cdot \overset{\mathrm{L}}{\underset{A}{\otimes}} N : \mathsf{D}_{\mathrm{perf}}(A^e) \to \mathsf{D}_{\mathrm{perf}}(A)$. Let pN be a cofibrant replacement of N. Then

$$A^e \overset{\mathrm{L}}{\underset{A}{\otimes}} N \simeq A^e \otimes_A pN \simeq A \otimes pN.$$

In $\mathcal{C}(k)$, pN is homotopically equivalent to $\mathrm{H}(pN) := \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^n(pN)[n]$. Thus, there is an isomorphism in $\mathsf{D}(A)$ between $A \otimes_k pN$ and $A \otimes_k \mathrm{H}(pN)$. The dg A-module $A \otimes_k \mathrm{H}(pN)$ is perfect. Thus, by Proposition 1.3.10, the functor $\cdot \overset{\mathrm{L}}{\underset{A}{\otimes}} N$ preserves perfect modules. Since A is homologically smooth, A belongs to $\mathsf{D}_{\mathrm{perf}}(A^e)$. Then, $A \overset{\mathrm{L}}{\underset{A}{\otimes}} N \simeq N$ belongs to $\mathsf{D}_{\mathrm{perf}}(A)$. □

A similar argument leads to (see [TV07, Lemma 2.6])

Lemma 1.3.20. *If A is a proper algebra then $\mathsf{D}_{\mathrm{perf}}(A)$ is Ext-finite.*

1.3.4 Serre duality for perfect modules

In this subsection, we recall some facts concerning Serre duality for perfect modules over a dg algebra and give various forms of the Serre functor in this context. References are made to [BK89], [Gin05], [Shk07a].

Let us recall the definition of a Serre functor, [BK89].

Definition 1.3.21. Let $\mathcal{C}$ be a k-linear Ext-finite triangulated category. A Serre functor $S : \mathcal{C} \to \mathcal{C}$ is an autoequivalence of $\mathcal{C}$ such that there exist an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(Y, X)^* \simeq \mathrm{Hom}_{\mathcal{C}}(X, S(Y)) \tag{1.3.2}$$

functorial with respect to X and Y where $*$ denote the dual with respect to k. If it exists, such a functor is unique up to natural isomorphism.

Notation 1.3.22. We set $\mathbb{D}'_A = \mathrm{RHom}_A(\cdot, A) : (\mathsf{D}(A))^{\mathrm{op}} \to \mathsf{D}(A^{\mathrm{op}})$.

Proposition 1.3.23. *The functor $\mathbb{D}'_A$ preserves perfect modules and induces an equivalence $(\mathsf{D}_{\mathrm{perf}}(A))^{\mathrm{op}} \to \mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}})$. When restricted to perfect modules, $\mathbb{D}'_{A^{\mathrm{op}}} \circ \mathbb{D}'_A \simeq \mathrm{id}$.*

Proof. See [Shk07b, Proposition A.1]. □

Proposition 1.3.24. *Suppose N is a perfect A-module and M is an arbitrary left $A\otimes B^{\mathrm{op}}$-module, where B is another dg algebra. Then there is a natural isomorphism of B-modules*

$$(\mathrm{RHom}_A(N,M))^* \simeq (M^*)^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A}{\otimes}} N \tag{1.3.3}$$

Theorem 1.3.25. *In $\mathsf{D}_{\mathrm{perf}}(A)$, the endofunctor $\mathrm{RHom}_A(\cdot,A)^*$ is isomorphic to the endofunctor $(A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} -$.*

Proof. This result is a direct corollary of Proposition 1.3.24 by choosing $M = A$ and $B = A$. ☐

Lemma 1.3.26 and Theorem 1.3.27 are probably well known results. Since we do not know any references for them, we shall give detailed proofs.

Lemma 1.3.26. *Let B be a proper dg algebra, $M \in \mathsf{D}_{\mathrm{perf}}(A\otimes B^{\mathrm{op}})$ and $N \in \mathsf{D}_{\mathrm{perf}}(B\otimes C^{\mathrm{op}})$. There are the following canonical isomorphisms respectively in $\mathsf{D}_{\mathrm{perf}}(B^{\mathrm{op}}\otimes C)$ and $\mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}}\otimes C)$:*

$$B^* \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \mathrm{RHom}_{B\otimes C^{\mathrm{op}}}(N, B\otimes C^{\mathrm{op}}) \simeq \mathrm{RHom}_{C^{\mathrm{op}}}(N, C^{\mathrm{op}}) \tag{1.3.4}$$

$$\begin{aligned}&\mathrm{RHom}_{A\otimes B^{\mathrm{op}}}(M, A\otimes \mathrm{RHom}_{C^{\mathrm{op}}}(N, C^{\mathrm{op}}))\\ &\qquad\qquad \simeq \mathrm{RHom}_{A\otimes C^{\mathrm{op}}}(M \overset{\mathrm{L}}{\underset{B}{\otimes}} N, A\otimes C^{\mathrm{op}}).\end{aligned} \tag{1.3.5}$$

Proof. *(i)* Let us prove formula (1.3.4). Let $N \in \mathcal{C}(B\otimes C^{\mathrm{op}})$. There is a morphism of $B^{\mathrm{op}}\otimes C$ modules

$$\Psi_N : B^* \otimes_{B^{\mathrm{op}}} \mathrm{Hom}^\bullet_{B\otimes C^{\mathrm{op}}}(N, B\otimes C^{\mathrm{op}}) \to \mathrm{Hom}^\bullet_{C^{\mathrm{op}}}(N, C^{\mathrm{op}})$$

such that $\Psi_N(\delta\otimes_{B^{\mathrm{op}}}\phi) = m\circ(\delta\otimes \mathrm{id}_{C^{\mathrm{op}}})\circ\phi$ where $m : k\otimes C^{\mathrm{op}} \to C^{\mathrm{op}}$ and $m(\lambda\otimes c) = \lambda\cdot c$. Clearly, Ψ is a natural transformation between the functor $B^*\otimes_{B^{\mathrm{op}}} \mathrm{Hom}^\bullet_{B\otimes C^{\mathrm{op}}}(\cdot, B\otimes C^{\mathrm{op}})$ and $\mathrm{Hom}^\bullet_{C^{\mathrm{op}}}(\cdot, C^{\mathrm{op}})$.

For short, we set

$$F(X) = B^* \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \mathrm{RHom}_{B\otimes C^{\mathrm{op}}}(X, B\otimes C^{\mathrm{op}}) \quad\text{and}\quad G(X) = \mathrm{RHom}_{C^{\mathrm{op}}}(X, C^{\mathrm{op}}).$$

If X is a direct factor of a finitely generated semi-free $B\otimes C^{\mathrm{op}}$-module, we obtain that $\mathrm{RHom}_{B\otimes C^{\mathrm{op}}}(X, B\otimes C^{\mathrm{op}}) \simeq \mathrm{Hom}^\bullet_{B\otimes C^{\mathrm{op}}}(X, B\otimes C^{\mathrm{op}})$ and the $B^{\mathrm{op}}\otimes C$-module $\mathrm{Hom}^\bullet_{B\otimes C^{\mathrm{op}}}(X, B\otimes C^{\mathrm{op}})$ is flat over B^{op} since it is flat over $B^{\mathrm{op}}\otimes C$. By Lemma 3.4.2 of [Hin97] we can use flat replacements instead of cofibrant one to compute derived tensor products. Thus $F(X) \simeq B^*\otimes_{B^{\mathrm{op}}} \mathrm{Hom}_{B\otimes C^{\mathrm{op}}}(X, B\otimes C^{\mathrm{op}})$.

Since $B\otimes C^{\mathrm{op}}$ is a cofibrant C^{op}-module, it follows that the forgetful functor from $\mathcal{C}(B\otimes C^{\mathrm{op}})$ to $\mathcal{C}(C^{\mathrm{op}})$ preserves cofibrations. Thus, X is a cofibrant C^{op}- module. It follows that $G(X) \simeq \mathrm{Hom}_{C^{\mathrm{op}}}(X, C^{\mathrm{op}})$. Therefore, Ψ induces a natural transformation from F to G when they are restricted to $\mathsf{D}_{\mathrm{perf}}(B\otimes C^{\mathrm{op}})$.

Assume that $X = B \otimes C^{\mathrm{op}}$. Then we have the following commutative diagram

$$\begin{tikzcd} B^* \otimes_{B^{\mathrm{op}}} \mathrm{Hom}^\bullet_{B\otimes C^{\mathrm{op}}}(B\otimes C^{\mathrm{op}}, B\otimes C^{\mathrm{op}}) \arrow[r, "\Psi_{B\otimes C^{\mathrm{op}}}"] \arrow[d, "\wr"] & \mathrm{Hom}^\bullet_{C^{\mathrm{op}}}(B\otimes C^{\mathrm{op}}, C^{\mathrm{op}}) \arrow[d, "\wr"] \\ B^* \otimes_{B^{\mathrm{op}}} B^{\mathrm{op}} \otimes C \arrow[d, "\wr"] & \mathrm{Hom}^\bullet_k(B, \mathrm{Hom}^\bullet_{C^{\mathrm{op}}}(C^{\mathrm{op}}, C^{\mathrm{op}}) \arrow[d, "\wr"] \\ B^* \otimes C \arrow[r, "\sim"] & \mathrm{Hom}^\bullet_k(B, C) \end{tikzcd}$$

which proves that $\Psi_{B\otimes C^{\mathrm{op}}}$ is an isomorphism. The bottom map of the diagram is an isomorphism because B is proper. Hence, by Proposition 1.3.11, Ψ_X is an isomorphism for any X in $\mathsf{D}_{\mathrm{perf}}(B \otimes C^{\mathrm{op}})$ which proves the claim.

(ii) Let us prove formula (1.3.5). We first notice that there is a morphism of $A \otimes C^{\mathrm{op}}$-modules functorial in M and N

$$\Theta : \mathrm{Hom}^\bullet_{A\otimes B^{\mathrm{op}}}(M, A \otimes \mathrm{Hom}^\bullet_{C^{\mathrm{op}}}(N, C^{\mathrm{op}})) \to \mathrm{Hom}^\bullet_{A\otimes C^{\mathrm{op}}}(M \otimes_B N, A \otimes C^{\mathrm{op}})$$

defined by $\psi \mapsto (\Psi : m \otimes n \mapsto \psi(m)(n))$.
If $M = A \otimes B^{\mathrm{op}}$ and $N = B \otimes C^{\mathrm{op}}$, then it induces an isomorphism. By applying an argument similar to the previous one we are able to establish the isomorphism (1.3.5). □

The next relative duality theorem can be compared to [KS12, Thm 3.3.3] in the framework of DQ-modules though the proof is completely different.

Theorem 1.3.27. *Assume that B is proper. Let $M \in \mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}})$ and $N \in \mathsf{D}_{\mathrm{perf}}(B \otimes C^{\mathrm{op}})$. There is a natural isomorphism in $\mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}} \otimes C)$*

$$\mathbb{D}'_{A\otimes B^{\mathrm{op}}}(M) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} B^* \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \mathbb{D}'_{B\otimes C^{\mathrm{op}}}(N) \simeq \mathbb{D}'_{A\otimes C^{\mathrm{op}}}(M \overset{\mathrm{L}}{\underset{B}{\otimes}} N).$$

Proof. We have

$$\begin{aligned} &\mathbb{D}'(M) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} B^* \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \mathbb{D}'(N) \\ &\qquad \simeq \mathrm{RHom}_{A\otimes B^{\mathrm{op}}}(M, A\otimes B^{\mathrm{op}}) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} B^* \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \mathrm{RHom}_{B\otimes C^{\mathrm{op}}}(N, B\otimes C^{\mathrm{op}}) \\ &\qquad \simeq \mathrm{RHom}_{A\otimes B^{\mathrm{op}}}(M, A\otimes B^{\mathrm{op}}) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \mathrm{RHom}_{C^{\mathrm{op}}}(N, C^{\mathrm{op}}) \\ &\qquad \simeq \mathrm{RHom}_{A\otimes B^{\mathrm{op}}}(M, A\otimes \mathrm{RHom}_{C^{\mathrm{op}}}(N, C^{\mathrm{op}})) \\ &\qquad \simeq \mathrm{RHom}_{A\otimes C^{\mathrm{op}}}(M \overset{\mathrm{L}}{\underset{B}{\otimes}} N, A\otimes C^{\mathrm{op}}). \end{aligned}$$

□

One has (see for instance [Gin05])

Theorem 1.3.28. *Let A be a proper homologically smooth dg algebra. The functor $N \mapsto (A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} N$, $\mathsf{D}_{\mathrm{perf}}(A) \to \mathsf{D}_{\mathrm{perf}}(A)$ is a Serre functor.*

Proof. According to Lemma 1.3.20, $\mathsf{D}_{\mathrm{perf}}(A)$ is an Ext-finite category. By Theorem 1.3.25, the functor $(A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} -$ is isomorphic to the functor $\mathrm{RHom}_A(\cdot, A)^*$. Moreover using Theorem 1.3.19 and Proposition 1.3.23 one sees that $\mathrm{RHom}_A(\cdot, A)^*$ is an equivalence on $\mathsf{D}_{\mathrm{perf}}(A)$ and so is the functor $(A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} \cdot$. By applying Theorem 1.3.27 with $A = C = k$, $B = A^{\mathrm{op}}$ $N = M$ and $M = \mathrm{RHom}_A(N, A)$ one obtains

$$\mathrm{RHom}_A(N, (A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} M) \simeq \mathrm{RHom}_A(M, N)^*.$$

□

Definition 1.3.29. One sets $S_A : \mathsf{D}_{\mathrm{perf}}(A) \to \mathsf{D}_{\mathrm{perf}}(A)$, $N \mapsto (A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} N$ for the Serre functor of $\mathsf{D}_{\mathrm{perf}}(A)$.

The Serre functor can also be expressed in term of dualizing objects. They are defined by [KS09], [vdB97], [Gin05]. Related results can also be found in [Jør04]. One sets:

$$\omega_A^{-1} := \mathrm{RHom}_{{}^eA}(A^{\mathrm{op}}, {}^eA) = \mathbb{D}'_{{}^eA}(A^{\mathrm{op}}) \qquad \text{and} \qquad \omega_A := \mathrm{RHom}_A(\omega_A^{-1}, A) = \mathbb{D}'_A(\omega_A^{-1}).$$

The structure of A^e-module of ω_A^{-1} is clear. The object ω_A inherits a structure of A^{op}-module from the structure of A^{op}-module of A and a structure of A-module from the structure of A^{op}-module of ω_A^{-1}. This endows ω_A with a structure of A^e-modules.

Since A is a smooth dg algebra, it is a perfect A^e-module. Proposition 1.3.23 ensures that ω_A^{-1} is a perfect A^e-module. Finally, Proposition 1.3.19 shows that ω_A is a perfect A^e-module.

Proposition 1.3.30. *The functor $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} -$ is left adjoint to the functor $\omega_A \overset{\mathrm{L}}{\underset{A}{\otimes}} -$.*

One also has, [Gin05]

Theorem 1.3.31. *The two functors $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} -$ and S_A from $\mathsf{D}_{\mathrm{perf}}(A)$ to $\mathsf{D}_{\mathrm{perf}}(A)$ are quasi-inverse.*

Proof. The functor S_A is an autoequivalence. Thus, it is a right adjoint of its inverse. We prove that $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} -$ is a left adjoint to S_A. On the one hand we have for every $N, M \in \mathsf{D}_{\mathrm{perf}}(A)$ the isomorphism

$$\mathrm{Hom}_{\mathsf{D}_{\mathrm{perf}}(A)}(N, S_A(M)) \simeq (\mathrm{Hom}_{\mathsf{D}_{\mathrm{perf}}(A)}(M, N))^*.$$

On the other hand we have the following natural isomorphisms

$$\begin{aligned}
\mathrm{RHom}_A(\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} N, M) &\simeq (M^* \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} N)^* \\
&\simeq ((\omega_A^{-1})^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} (N \otimes M^*))^* \\
&\simeq \mathrm{RHom}_{A^e}(\mathrm{RHom}_{{}^eA}(\omega_{A^{\mathrm{op}}}^{-1}, {}^eA), (N \otimes M^*))^* \\
&\simeq \mathrm{RHom}_{A^e}(A, (N \otimes M^*))^*.
\end{aligned}$$

Using the isomorphism

$$\mathrm{Hom}_{\mathsf{D}_{\mathrm{perf}}(A^e)}(A, N \otimes M^*) \simeq \mathrm{Hom}_{\mathsf{D}_{\mathrm{perf}}(A)}(M, N),$$

we obtain the desired result. □

Corollary 1.3.32. *The functors* $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} \cdot : \mathsf{D}_{\mathrm{perf}}(A) \to \mathsf{D}_{\mathrm{perf}}(A)$ *and* $\omega_A \overset{\mathrm{L}}{\underset{A}{\otimes}} \cdot : \mathsf{D}_{\mathrm{perf}}(A) \to \mathsf{D}_{\mathrm{perf}}(A)$ *are equivalences of categories.*

Corollary 1.3.33. *The natural morphisms in* $\mathsf{D}_{\mathrm{perf}}(A^e)$

$$\begin{aligned} A &\to \mathrm{RHom}_A(\omega_A^{-1}, \omega_A^{-1}) \\ A &\to \mathrm{RHom}_A(\omega_A, \omega_A) \end{aligned}$$

are isomorphisms.

Proof. The functor $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} \cdot$ induces a morphism in $\mathsf{D}(A^e)$

$$A \simeq \mathrm{RHom}_A(A, A) \to \mathrm{RHom}_A(\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} A, \omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} A) \simeq \mathrm{RHom}_A(\omega_A^{-1}, \omega_A^{-1}).$$

Since $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} \cdot$ is an equivalence of category, for every $i \in \mathbb{Z}$

$$\mathrm{Hom}_A(A, A[i]) \xrightarrow{\sim} \mathrm{Hom}_A(\omega_A^{-1}, \omega_A^{-1}).$$

The results follows immediately. The proof is similar for the second morphism. □

Proposition 1.3.34. *Let A be a proper homologically smooth dg algebra. We have the isomorphisms of A^e-modules*

$$\omega_A \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A^{-1} \simeq A, \qquad \omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A \simeq A.$$

Proof. We have

$$\begin{aligned} \omega_A \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A^{-1} &\simeq \mathrm{RHom}_A(\omega_A^{-1}, A) \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A^{-1} \\ &\simeq \mathrm{RHom}_A(\omega_A^{-1}, \omega_A^{-1}) \\ &\simeq A. \end{aligned}$$

For the second isomorphism, we remark that

$$\begin{aligned} \mathrm{RHom}_A(\omega_A, A) \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A &\simeq \mathrm{RHom}_A(\omega_A, \omega_A) \\ &\simeq A, \end{aligned}$$

and

$$\begin{aligned} \mathrm{RHom}_A(\omega_A, A) &\simeq \mathrm{RHom}_A(\omega_A, A) \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A^{-1} \\ &\simeq \omega_A^{-1} \end{aligned}$$

which conclude the proof. □

Corollary 1.3.35. *Let A be a proper homologically smooth dg algebra. The two objects $(A^{\mathrm{op}})^*$ and ω_A of $\mathsf{D}_{\mathrm{perf}}(A^e)$ are isomorphic.*

Proof. Applying Theorem 1.3.27 with $A = B = C = A^{\mathrm{op}}$, $M = N = A^{\mathrm{op}}$, we get that $\omega_A^{-1} \overset{\mathrm{L}}{\underset{A}{\otimes}} (A^{\mathrm{op}})^* \overset{\mathrm{L}}{\underset{A}{\otimes}} \omega_A^{-1} \simeq \omega_A^{-1}$ in $\mathsf{D}_{\mathrm{perf}}(A^e)$. Then, the result follows from Corollary 1.3.34. □

Remark 1.3.36. Since ω_A and $(A^{\mathrm{op}})^*$ are isomorphic as A^e-modules, we will use ω_A to denote both A^* and $\mathrm{RHom}_A(\omega_A^{-1}, A)$ considered as the dualizing complexes of the category $\mathsf{D}_{\mathrm{perf}}(A)$.

The previous results allow us to build an "integration" morphism.

Proposition 1.3.37. *There exists a natural "integration" morphism in* $\mathsf{D}_{\mathrm{perf}}(k)$

$$\omega_{A^{\mathrm{op}}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \to k.$$

Proof. There is a natural morphism $k \to \mathrm{RHom}_{A^e}(A, A)$. Applying $(\cdot)^*$ and formula (1.3.3) with $A = A^e$ and $B = k$, we obtain a morphism $A^* \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \to k$. Here, A^* is endowed with its standard structure of right A^e-modules that is to say with its standard structure of left ${}^eA$-module. Thus, $\omega_{A^{\mathrm{op}}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \simeq A^* \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \to k$. □

Corollary 1.3.38. *There exists a canonical map* $\omega_A \to k$ *in* $\mathsf{D}_{\mathrm{perf}}(k)$ *induced by the morphism of Proposition 1.3.37.*

1.4 Hochschild homology and Hochschild classes

1.4.1 Hochshchild homology

In this subsection we recall the definition of Hochschild homology, (see [Kel98a], [Lod98]) and prove that it can be expressed in terms of dualizing objects, (see [Căl03], [CW10], [KS12], [KS09]).

Definition 1.4.1. The Hochshchild homology of a dg algebra is defined by

$$\mathcal{HH}(A) := A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A.$$

The Hochschild homology groups are defined by $\mathrm{HH}_n(A) = \mathrm{H}^{-n}(A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A)$.

Proposition 1.4.2. *If A is a proper and homologically smooth dg algebra then there is a natural isomorphism*

$$\mathcal{HH}(A) \simeq \mathrm{RHom}_{A^e}(\omega_A^{-1}, A). \tag{1.4.1}$$

Proof. We have

$$\begin{aligned} A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A &\simeq (\mathbb{D}'_{A^e} \circ \mathbb{D}'_{{}^eA}(A^{\mathrm{op}})) \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \\ &\simeq \mathbb{D}'_{A^e}(\omega_A^{-1}) \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \\ &\simeq \mathrm{RHom}_{A^e}(\omega_A^{-1}, A). \end{aligned}$$

□

Remark 1.4.3. There is also a natural isomorphism

$$\mathcal{HH}(A) \simeq \mathrm{RHom}_{A^e}(A, \omega_A).$$

It is obtain by adjunction from isomorphism (1.4.1).

There is the following natural isomorphism.

Proposition 1.4.4. *Let A and B be a dg algebras. Let $M \in \mathsf{D}_{\mathrm{perf}}(A)$ and $S \in \mathsf{D}_{\mathrm{perf}}(B)$ and $N \in \mathsf{D}(A)$ and $T \in \mathsf{D}(B)$ then*

$$\mathrm{RHom}_{A\otimes B}(M \otimes S, N \otimes T) \simeq \mathrm{RHom}_A(M, N) \otimes \mathrm{RHom}_B(S, T).$$

Proof. clear. □

A special case of the above proposition is

Proposition 1.4.5 (Künneth isomorphism)**.** *Let A and B be proper homologically smooth dg algebras. There is a natural isomorphism*

$$\mathfrak{K}_{A,B} : \mathcal{HH}(A) \otimes \mathcal{HH}(B) \xrightarrow{\sim} \mathcal{HH}(A \otimes B).$$

1.4.2 The Hochschild class

In this subsection, following [KS12], we construct the Hochschild class of an endomorphism of a perfect module and describe the Hochschild class of this endomorphism when the Hochschild homology is expressed in term of dualizing objects.

To build the Hochschild class, we need to construct some morphism of $\mathsf{D}_{\mathrm{perf}}(A^e)$.

Lemma 1.4.6. *Let M be a perfect A-module. There is a natural isomorphism*

$$\mathrm{RHom}_A(M, M) \xrightarrow{\sim} \mathrm{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}'_A M). \tag{1.4.2}$$

Proof. We have

$$\begin{aligned}\mathrm{RHom}_A(M, M) &\simeq \mathbb{D}'_A M \overset{\mathrm{L}}{\underset{A}{\otimes}} M \\ &\simeq A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} (M \otimes \mathbb{D}'_A M) \\ &\simeq \mathrm{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}'_A M).\end{aligned}$$

Thus, we get an isomorphism

$$\mathrm{RHom}_A(M, M) \xrightarrow{\sim} \mathrm{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}'_A M). \tag{1.4.3}$$

□

Definition 1.4.7. The morphism η in $\mathsf{D}_{\mathrm{perf}}(A^e)$ is the image of the identity of M by morphism (1.4.2) and ε in $\mathsf{D}_{\mathrm{perf}}(A^e)$ is obtained from η by duality.

$$\eta : \omega_A^{-1} \to M \otimes \mathbb{D}'_A M, \tag{1.4.4}$$

$$\varepsilon : M \otimes \mathbb{D}'_A M \to A. \tag{1.4.5}$$

The map η is called the coevaluation map and ε the evaluation map.

Applying $\mathbb{D}'_{A^e}$ to (1.4.4) we obtain a map

$$\mathbb{D}'_{A^e}(M \otimes \mathbb{D}'_A M) \to A^{\mathrm{op}}.$$

Then using the isomorphism of Proposition 1.4.4,

$$\begin{aligned}\mathbb{D}'_{A^e}(M \otimes \mathbb{D}'_A M) &\simeq \mathbb{D}'_A(M) \otimes \mathbb{D}'_{A^{\mathrm{op}}}(\mathbb{D}'_A M)\\ &\simeq \mathbb{D}'_A(M) \otimes M,\end{aligned}$$

we get morphism (1.4.5).

Let us define the Hochschild class. We have the following chain of morphisms

$$\begin{aligned}\mathrm{RHom}_A(M,M) &\simeq \mathbb{D}'_A M \overset{\mathrm{L}}{\underset{A}{\otimes}} M\\ &\simeq A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} (M \otimes \mathbb{D}'_A M)\\ &\xrightarrow{\mathrm{id}\otimes\varepsilon} A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A.\end{aligned}$$

We get a map

$$\mathrm{Hom}_{\mathsf{D}_{\mathrm{perf}}(A)}(M,M) \to \mathrm{HH}_0(A). \tag{1.4.6}$$

Definition 1.4.8. The image of an element $f \in \mathrm{Hom}_{\mathsf{D}_{\mathrm{perf}}(A)}(M,M)$ by the map (1.4.6) is called the Hochschild class of f and is denoted $\mathrm{hh}_A(M,f)$. The Hochschild class of the identity is denoted $\mathrm{hh}_A(M)$ and is called the Hochschild class of M.

Remark 1.4.9. If $A = k$ and $M \in \mathsf{D}_{\mathrm{perf}}(k)$, then

$$\mathrm{hh}_k(M,f) = \sum_i (-1)^i \,\mathrm{Tr}(\mathrm{H}^i(f : M \to M)),$$

see for instance [KS12].

Lemma 1.4.10. *The isomorphism (1.4.1) sends* $\mathrm{hh}_A(M,f)$ *to the image of* f *under the composition*

$$\mathrm{RHom}_A(M,M) \xrightarrow{\sim} \mathrm{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}'_A M) \xrightarrow{\varepsilon\circ} \mathrm{RHom}_{A^e}(\omega_A^{-1}, A)$$

where the first morphism is defined in (1.4.2) and the second morphism is induced by the evaluation map.

Proof. This follows from the commutative diagram:

$$\begin{tikzcd}
\mathrm{RHom}_A(M,M) \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"'] & A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} (M \otimes \mathbb{D}'_A M) \arrow[r, "\mathrm{id}_A \otimes\varepsilon"] \arrow[d] & A^{\mathrm{op}} \overset{\mathrm{L}}{\underset{A^e}{\otimes}} A \arrow[d] \\
\mathrm{RHom}_A(M,M) \arrow[r, "\sim"] & \mathrm{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}'_A M) \arrow[r, "\varepsilon"] & \mathrm{RHom}_{A^e}(\omega_A^{-1}, A).
\end{tikzcd}$$

□

Remark 1.4.11. Our definition of the Hochschild class is equivalent to the definition of the trace of a 2-cell in [PS11]. This equivalence allows us to use string diagrams to prove some properties of the Hochschild class, see [PS11] and [Căl03], [CW10].

Proposition 1.4.12. *Let* $M, N \in \mathsf{D}_{\mathrm{perf}}(A)$*,* $g \in \mathrm{Hom}_A(M,N)$ *and* $h \in \mathrm{Hom}_A(N,M)$ *then*

$$\mathrm{hh}_A(N, g\circ h) = \mathrm{hh}_A(M, h \circ g).$$

Proof. See for instance [PS11, §7]. □

1.5 A pairing on Hochschild homology

In this section, we build a pairing on Hochschild homology. It acts as the Hochschild class of the diagonal, (see [KS12], [Căl03], [CW10], [Shk07a]). Using this result, we prove our Riemann-Roch type formula. We follow the approach of [KS12].

1.5.1 Hochschild homology and bimodules

In this subsection, we translate to our language the classical fact that a perfect $A \otimes B^{\mathrm{op}}$-module induces a morphism from $\mathcal{HH}(B)$ to $\mathcal{HH}(A)$. We need the following technical lemma which generalizes Lemma 1.4.6.

Lemma 1.5.1. *Let $K \in \mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}})$. Let $C = A \otimes B^{\mathrm{op}}$. Then, there are natural morphisms in $\mathsf{D}_{\mathrm{perf}}(A^e)$ which coincide with (1.4.4) and (1.4.5) when $B = k$,*

$$\omega_A^{-1} \to K \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_C K,$$

$$K \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_C K \to A. \tag{1.5.1}$$

Proof. By (1.4.4), we have a morphism in $\mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}} \otimes B \otimes A^{\mathrm{op}})$

$$\omega_C^{-1} \to K \otimes \mathbb{D}'_C K.$$

Applying the functor $- \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B$, we obtain

$$\omega_C^{-1} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \to K \otimes \mathbb{D}'_C K \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B$$

and by Proposition 1.4.4

$$\omega_C^{-1} \simeq \omega_A^{-1} \otimes \omega_{B^{\mathrm{op}}}^{-1} \simeq \omega_A^{-1} \otimes \mathbb{D}'_{B^e}(B).$$

Then there is a sequence of isomorphisms

$$\omega_A^{-1} \otimes \mathrm{RHom}_{B^e}(B, B) \xrightarrow{\sim} \omega_A^{-1} \otimes (\mathbb{D}'_{B^e} B \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B) \xrightarrow{\sim} \omega_C^{-1} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B$$

and there is a natural arrow $\omega_A^{-1} \xrightarrow{\mathrm{id} \otimes 1} \omega_A^{-1} \otimes \mathrm{RHom}_{B^e}(B, B)$. Composing these maps, we obtain the morphism

$$\omega_A^{-1} \to \omega_A^{-1} \otimes (\mathbb{D}'_{B^e} B \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B) \to \omega_C^{-1} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \to (K \otimes \mathbb{D}'_C K) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B.$$

For the map (1.5.1), we have a morphism in $\mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}} \otimes B \otimes A^{\mathrm{op}})$ given by the map (1.4.5)

$$K \otimes \mathbb{D}'_C K \to C.$$

Then applying the functor $- \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B$, we obtain

$$(K \otimes \mathbb{D}'_C K) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B \to C \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B \simeq (A \otimes B^{\mathrm{op}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B.$$

Composing with the natural "integration" morphism of Proposition 1.3.37, we get

$$(K \otimes \mathbb{D}'_C K) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B \to A$$

which proves the lemma. □

We shall show that an object in $\mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}})$ induces a morphism between the Hochschild homology of A and that of B.

Let $K \in \mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}})$. We set $C = A \otimes B^{\mathrm{op}}$ and $S = K \otimes (\omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_C(K)) \in \mathsf{D}(A^e \otimes (B^e)^{\mathrm{op}})$.

We have

$$S \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B^{-1} \simeq K \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B^{-1} \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}_C(K) \qquad S \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \simeq K \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_C(K).$$
$$\simeq K \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_C(K)$$

The map

$$\Phi_K : \mathcal{HH}(B) \to \mathcal{HH}(A). \tag{1.5.2}$$

is defined as follow.

$$\begin{aligned}\mathrm{RHom}_{B^e}(\omega_B^{-1}, B) &\to \mathrm{RHom}_C(S \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B^{-1}, S \overset{\mathrm{L}}{\underset{B}{\otimes}} B)\\ &\to \mathrm{RHom}_{A^e}(\omega_A^{-1}, A).\end{aligned}$$

The last arrow is associated with the morphisms in Lemma 1.5.1. This defines the map (1.5.2).

1.5.2 A pairing on Hochschild homology

In this subsection, we build a pairing on the Hochschild homology of a dg algebra. It acts as the Hochschild class of the diagonal, (see [KS12], [Căl03], [CW10], [Shk07a]). We also relate Φ_K to this pairing.

A natural construction to obtain a pairing on the Hochshild homology of a dg algebra A is the following one.

Consider A as a perfect $k - {}^eA$ bimodule. The morphism (1.5.2) with $K = A$ provides a map

$$\Phi_A : \mathcal{HH}({}^eA) \to \mathcal{HH}(k).$$

We compose Φ_A with $\mathfrak{K}_{A^{\mathrm{op}},A}$ and get

$$\mathcal{HH}(A^{\mathrm{op}}) \otimes \mathcal{HH}(A) \to k. \tag{1.5.3}$$

Taking the 0^th degree homology, we obtain

$$\langle \cdot, \cdot \rangle : (\bigoplus_{n \in \mathbb{Z}} \mathrm{HH}_{-n}(A^{\mathrm{op}}) \otimes \mathrm{HH}_n(A)) \to k. \tag{1.5.4}$$

In other words $\langle \cdot, \cdot \rangle = \mathrm{H}^0(\Phi_A) \circ \mathrm{H}^0(\mathfrak{K}_{A^{\mathrm{op}},A})$.

However, it is not clear how to express Φ_K in term of the Hochschild class of K using the above construction of the pairing. Thus, we propose another construction of the pairing and shows it coincides with the previous one.

Proposition 1.5.2. *Let A, B, C be three proper homologically smooth dg algebras and K an object of $\mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}})$.*
There is a natural map

$$\mathcal{HH}(A \otimes B^{\mathrm{op}}) \otimes \mathcal{HH}(B \otimes C^{\mathrm{op}}) \to \mathcal{HH}(A \otimes C^{\mathrm{op}})$$

inducing, for every $i \in \mathbb{Z}$, an operation

$$\cup_B : \bigoplus_{n\in\mathbb{Z}} (\mathrm{HH}_{-n}(A \otimes B^{\mathrm{op}}) \otimes \mathrm{HH}_{n+i}(B \otimes C^{\mathrm{op}})) \to \mathrm{HH}_i(A \otimes C^{\mathrm{op}})$$

such that for every $\lambda \in \mathrm{HH}_i(B \otimes C^{\mathrm{op}})$, $\mathrm{H}^i(\Phi_K \otimes \mathrm{id})(\lambda) = \mathrm{hh}_{A\otimes B^{\mathrm{op}}}(K) \cup_B \lambda$.

Before proving Proposition 1.5.2, let us do the following remark.

Remark 1.5.3. Let $M \in \mathsf{D}_{\mathrm{perf}}(A)$. There is an isomorphism in $\mathsf{D}_{\mathrm{perf}}(k)$

$$\omega_A \overset{\mathrm{L}}{\underset{A}{\otimes}} M \simeq M \overset{\mathrm{L}}{\underset{A^{\mathrm{op}}}{\otimes}} \omega_{A^{\mathrm{op}}}.$$

The next proof explains the construction of $\cup_B : \bigoplus_{n\in\mathbb{Z}} (\mathrm{HH}_{-n}(A \otimes B^{\mathrm{op}}) \otimes \mathrm{HH}_{n+i}(B \otimes C^{\mathrm{op}})) \to \mathrm{HH}_i(A \otimes C^{\mathrm{op}})$. We also prove the equality $\mathrm{H}^i(\Phi_K \otimes \mathrm{id})(\lambda) = \mathrm{hh}_{A\otimes B^{\mathrm{op}}}(K) \cup_B \lambda$.

Proof of Proposition 1.5.2 . (i) We identify $(A \otimes B^{\mathrm{op}})^{\mathrm{op}}$ and $A^{\mathrm{op}} \otimes B$.
We have

$$\begin{aligned}
\mathcal{HH}(A \otimes B^{\mathrm{op}}) &\simeq \mathrm{RHom}_{A^e \otimes^e B}(\omega^{-1}_{A\otimes B^{\mathrm{op}}}, A \otimes B^{\mathrm{op}}) \\
&\simeq \mathrm{RHom}_{A^e \otimes^e B}(\omega_A^{-1} \otimes \omega^{-1}_{B^{\mathrm{op}}} \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}, A \otimes B^{\mathrm{op}} \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \\
&\simeq \mathrm{RHom}_{A^e \otimes^e B}(\omega_A^{-1} \otimes B^{\mathrm{op}}, A \otimes \omega_{B^{\mathrm{op}}}).
\end{aligned}$$

Let $S_{AB} = \omega_A^{-1} \otimes B^{\mathrm{op}}$ and $T_{AB} = A \otimes \omega_{B^{\mathrm{op}}}$. Similarly, we define S_{BC} and T_{BC}. Then, we get

$$\begin{aligned}
\mathcal{HH}(A \otimes B^{\mathrm{op}}) &\otimes \mathcal{HH}(B \otimes C^{\mathrm{op}}) \\
&\simeq \mathrm{RHom}_{A^e \otimes^e B}(S_{AB}, T_{AB}) \otimes \mathrm{RHom}_{B^e \otimes^e C}(S_{BC}, T_{BC}) \\
&\to \mathrm{RHom}_{A^e \otimes^e C}(S_{AB} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} S_{BC}, T_{AB} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} T_{BC}).
\end{aligned}$$

Using the morphism $k \to \mathrm{RHom}_{^eB}(B^{\mathrm{op}}, B^{\mathrm{op}})$, we get

$$k \to B^{\mathrm{op}} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B^{-1}.$$

Thus, we get

$$\omega_A^{-1} \otimes C^{\mathrm{op}} \to (\omega_A^{-1} \otimes B^{\mathrm{op}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} (\omega_B^{-1} \otimes C^{\mathrm{op}}).$$

We know by Proposition 1.3.37 that there is a morphism

$$\omega_{B^{\mathrm{op}}} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \to k.$$

we deduce a morphism

$$(A \otimes \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} (B \otimes \omega_{C^{\mathrm{op}}}) \to A \otimes \omega_{C^{\mathrm{op}}}.$$

Therefore we get

$$\mathcal{HH}(A \otimes B^{\mathrm{op}}) \otimes \mathcal{HH}(B \otimes C^{\mathrm{op}}) \to \mathcal{HH}(A \otimes C^{\mathrm{op}}).$$

Finally, taking the cohomology we obtain,

$$\bigoplus_{n\in\mathbb{Z}} (\mathrm{HH}_{-n}(A \otimes B^{\mathrm{op}}) \otimes \mathrm{HH}_{n+i}(B \otimes C^{\mathrm{op}})) \to \mathrm{HH}_i(A \otimes C^{\mathrm{op}}).$$

(ii) We follow the proof of [KS12]. We only need to prove the case where $C = k$. The general case being a consequences of Lemma 1.5.4 (i) below. We set $P = A \otimes B^{\mathrm{op}}$. Let $\alpha = \mathrm{hh}_{A\otimes B^{\mathrm{op}}}(K)$. We assume that $\lambda \in \mathrm{HH}_0(B)$. The proof being similar if $\lambda \in \mathrm{HH}_i(B)$. By Proposition 1.4.10, α can be viewed as a morphism of the form

$$\omega^{-1}_{A\otimes B^{\mathrm{op}}} \to K \otimes \mathbb{D}'_P(K) \to A \otimes B^{\mathrm{op}}.$$

We consider λ as a morphism $\omega_B^{-1} \to B$. Then, following the construction of Φ_K, we observe that $\Phi_K(\lambda)$ is obtained as the composition

$$\omega_A^{-1} \longrightarrow K \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K \xrightarrow{\lambda} K \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K \longrightarrow A$$

We have the following commutative diagram in $\mathsf{D}_{\mathrm{perf}}(k)$.

$$\begin{tikzcd}
\omega_A^{-1} \arrow[d] & & \\
(\omega_A^{-1} \otimes B^{\mathrm{op}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B^{-1} \arrow[r, "\lambda"] \arrow[d, "\wr"] & \omega_A \otimes B^{\mathrm{op}} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[d, "\wr"] & \\
((\omega_A^{-1} \otimes \omega_{B^{\mathrm{op}}}^{-1}) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B^{-1} \arrow[r, "\lambda"] \arrow[d, "\wr"] & ((\omega_A^{-1} \otimes \omega_{B^{\mathrm{op}}}^{-1}) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[d] \arrow[dr] & \\
(\omega_{A\otimes B}^{-1} \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B^{-1} \arrow[r, "\lambda"] \arrow[d] & (\omega_{A\otimes B^{\mathrm{op}}}^{-1} \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[d] & ((K \otimes \mathbb{D}'_P K) \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[dl, "\sim"'] \arrow[d] \\
(K \otimes \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} \omega_B^{-1} \arrow[r, "\lambda"] \arrow[d, "\wr"] & (K \otimes \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[d, "\wr"] & (A \otimes B^{\mathrm{op}} \overset{\mathrm{L}}{\underset{B^{\mathrm{op}}}{\otimes}} \omega_{B^{\mathrm{op}}}) \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[d, "\wr"] \\
K \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B^{-1} \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K \arrow[r, "\lambda"] & K \overset{\mathrm{L}}{\underset{B}{\otimes}} B \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K \arrow[d] & A \otimes \omega_{B^{\mathrm{op}}} \overset{\mathrm{L}}{\underset{B^e}{\otimes}} B \arrow[d] \\
& K \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_P K \arrow[r] & A.
\end{tikzcd}$$

This diagram is obtained by computing $\mathrm{H}^0(\Phi_K)(\lambda)$ and $\alpha \cup \lambda$. The left column and the row on the bottom induces $\mathrm{H}^0(\Phi_K)(\lambda)$ whereas the row on the top and the right column induces $\alpha \cup \lambda$. This diagram commutes, consequently $\mathrm{H}^0(\Phi_K)(\lambda) = \mathrm{hh}_{A\otimes B^{\mathrm{op}}}(K) \cup \lambda$.

$\square$

We now give some properties of this operation.

Lemma 1.5.4. *Let A, B, C and S be proper homologically smooth dg algebras, $\lambda_{AB^{\mathrm{op}}} \in \mathrm{HH}_i(A \otimes B^{\mathrm{op}})$. Then*

(i) $\cup_B \circ (\cup_A \otimes \mathrm{id}) = \cup_A \circ (\mathrm{id} \otimes \cup_B) = \cup_{A\otimes B^{\mathrm{op}}}$.

(ii) $\mathrm{hh}_{A\otimes A^{\mathrm{op}}}(A) \cup_A \lambda_{AB^{\mathrm{op}}} = \lambda_{AB^{\mathrm{op}}}$ *and* $\lambda_{AB^{\mathrm{op}}} \cup_B \mathrm{hh}_{B\otimes B^{\mathrm{op}}}(B) = \lambda_{AB^{\mathrm{op}}}$.

Proof. (i) is obtained by a direct computation using the definition of $\cup$

(ii) results from Proposition 1.5.2 *(ii)* by noticing that Φ_A and Φ_B are equal to the identity.

□

From this natural operation we are able to deduce a pairing on Hochschild homology. Indeed using Proposition 1.5.2 we obtain a pairing

$$\cup : \bigoplus_{n\in\mathbb{Z}} (\mathrm{HH}_{-n}(A^{\mathrm{op}}) \otimes \mathrm{HH}_n(A)) \to \mathrm{HH}_0(k) \simeq k. \tag{1.5.5}$$

To relate the two preceding constructions of the pairing, we introduce a third way to construct it. Proposition 1.5.2 gives us a map

$$\cup_{{}^eA} : \bigoplus_{n\in\mathbb{Z}} (\mathrm{HH}_{-n}(A^e) \otimes \mathrm{HH}_n({}^eA)) \to \mathrm{HH}_0(k) \simeq k.$$

Then there is a morphism

$$\begin{aligned} \mathrm{HH}_{-n}(A^{\mathrm{op}}) \otimes \mathrm{HH}_n(A) &\to k \\ \lambda \otimes \mu &\mapsto \mathrm{hh}_{A^e}(A) \cup_{{}^eA} (\lambda \otimes \mu). \end{aligned}$$

Using Proposition 1.5.2, we get that

$$\mathrm{H}^i(\Phi_A)(\lambda \otimes \mu) = \mathrm{hh}_{A^e}(A) \cup_{{}^eA} (\lambda \otimes \mu).$$

By Lemma 1.5.4, we have

$$\begin{aligned} \mathrm{hh}_{A^e}(A) \cup_{{}^eA} (\lambda \otimes \mu) &= (\lambda \cup_A \mathrm{hh}_{A\otimes A^{\mathrm{op}}}(A)) \cup_A \mu \\ &= \lambda \cup \mu. \end{aligned}$$

This proves that these three ways of defining a pairing lead to the same pairing. It also shows that the pairing is equivalent to the action of the Hochschild class of the diagonal.

1.5.3 Riemann-Roch formula for dg algebras

In this section we prove the Riemann-Roch formula announced in the introduction.

Proposition 1.5.5. *Let $M \in \mathsf{D}_{\mathrm{perf}}({}^eA)$ and let $f \in \mathrm{Hom}_A(M, M)$. Then*

$$\mathrm{hh}_k(A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} M, \mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f) = \mathrm{hh}_{A^e}(A) \cup \mathrm{hh}_{{}^eA}(M, f)$$

Proof. Let $\lambda = \mathrm{hh}_{{}^eA}(M,f) \in \mathrm{HH}_0({}^eA) \simeq \mathrm{Hom}(\omega_{{}^eA}^{-1}, {}^eA)$. As previously, we set $B = {}^eA$. We denote by $\tilde{f}$ the image of f in $\mathrm{Hom}_{B^e}(\omega_B^{-1}, M \otimes \mathbb{D}'_B M)$ by the isomorphism (1.4.2) and by $\widetilde{\mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f}$ the image of $\mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f$ by the isomorphism (1.4.2) applied with $A = k$ and $M = A \overset{\mathrm{L}}{\underset{B}{\otimes}} M$. We obtain the commutative diagram below.

$$\begin{tikzcd}
\omega_k^{-1} \arrow[r] \arrow[ddr, bend right, "\widetilde{\mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f}"'] & A \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B^{-1} \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \arrow[r, "\lambda"] \arrow[d, "\tilde{f}"] & A \overset{\mathrm{L}}{\underset{B}{\otimes}} B \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \arrow[r] & k \\
& A \overset{\mathrm{L}}{\underset{B}{\otimes}} (M \otimes \mathbb{D}'_B M) \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \arrow[ur, "\varepsilon"] \arrow[d, "\wr"] & & \\
& (A \overset{\mathrm{L}}{\underset{B}{\otimes}} M) \otimes \mathbb{D}'_k (A \overset{\mathrm{L}}{\underset{B}{\otimes}} M) \arrow[uurr, bend right] & &
\end{tikzcd}$$

The map $\omega_k^{-1} \to A \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B^{-1} \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A$ is obtained by applying Lemma 1.5.1 with $A = k$, $B = {}^eA$, $K = A$. Then,

$$\omega_k^{-1} \to A \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \simeq A \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B^{-1} \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A.$$

The morphism $A \overset{\mathrm{L}}{\underset{B}{\otimes}} B \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \to k$ is obtained as the composition of

$$A \overset{\mathrm{L}}{\underset{B}{\otimes}} B \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \simeq A \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A$$

with

$$A \overset{\mathrm{L}}{\underset{B}{\otimes}} \omega_B \overset{\mathrm{L}}{\underset{B}{\otimes}} \mathbb{D}'_{B^{\mathrm{op}}} A \to k. \tag{1.5.6}$$

The morphism (1.5.6) is the map (1.5.1) with $A = k$, $B = {}^eA$, $K = A$.

The vertical isomorphism is obtained by applying Theorem 1.3.27 with $A = C = k$, $B = B^{\mathrm{op}}$, $M = M$ and $N = A$.

By Lemma 1.4.10, the composition of the arrows on the bottom is $\mathrm{hh}_k(A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} M, \mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f)$ and the composition of the arrow on the top is $\mathrm{H}^0(\Phi_A(\mathrm{hh}_{{}^eA}(M,f)))$. It results from the commutativity of the diagram that

$$\mathrm{hh}_k(A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} M, \mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f) = \mathrm{H}^0(\Phi_A)(\mathrm{hh}_{{}^eA}(M,f)).$$

Then using Proposition 1.5.2 we get

$$\mathrm{hh}_k(A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} M, \mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} f) = \mathrm{hh}_{A^e}(A) \cup \mathrm{hh}_{{}^eA}(M,f).$$

□

We state and prove our main result which can be viewed as a noncommutative generalization of A. Căldăraru's version of the topological Cardy condition [Căl03].

Theorem 1.5.6. *Let $M \in \mathsf{D}_{\mathrm{perf}}(A)$, $f \in \mathrm{Hom}_A(M, M)$ and $N \in \mathsf{D}_{\mathrm{perf}}(A^{\mathrm{op}})$, $g \in \mathrm{Hom}_{A^{\mathrm{op}}}(N, N)$. Then*

$$\mathrm{hh}_k(N \overset{\mathrm{L}}{\underset{A}{\otimes}} M, g \overset{\mathrm{L}}{\underset{A}{\otimes}} f) = \mathrm{hh}_{A^{\mathrm{op}}}(N, g) \cup \mathrm{hh}_A(M, f).$$

where $\cup$ is the pairing defined by formula (1.5.5).

Proof. Let u be the canonical isomorphism from $A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} (N \otimes M)$ to $N \overset{\mathrm{L}}{\underset{A}{\otimes}} M$. By definition of the pairing we have

$$\begin{aligned}
\langle \mathrm{hh}_{A^{\mathrm{op}}}(N, g), \mathrm{hh}_A(M, f)\rangle &= \mathrm{H}^0(\Phi_A) \circ \mathrm{H}^0(\mathfrak{K}_{A^{\mathrm{op}}, A})(\mathrm{hh}_{A^{\mathrm{op}}}(N, g) \otimes \mathrm{hh}_A(M, f)) \\
&= \mathrm{hh}_{A^e}(A) \cup \mathrm{hh}_{{}^eA}(N \otimes M, g \otimes f) \\
&= \mathrm{hh}_k(A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} (N \otimes M), \mathrm{id}_A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} (g \otimes f)) \\
&= \mathrm{hh}_k(A \overset{\mathrm{L}}{\underset{{}^eA}{\otimes}} (N \otimes M), u^{-1} \circ (g \overset{\mathrm{L}}{\underset{A}{\otimes}} f) \circ u) \\
&= \mathrm{hh}_k(N \overset{\mathrm{L}}{\underset{A}{\otimes}} M, g \overset{\mathrm{L}}{\underset{A}{\otimes}} f).
\end{aligned}$$

The last equality is a consequence of Proposition 1.4.12. □

Remark 1.5.7. By adapting the proof of Proposition 1.5.5, we are also able to obtain the following result that should be compared to [KS12, Theorem 4.3.4]

Theorem 1.5.8. *Let A, B, C be proper homologically smooth dg algebras. Let $K_1 \in \mathsf{D}_{\mathrm{perf}}(A \otimes B^{\mathrm{op}})$, $K_2 \in \mathsf{D}_{\mathrm{perf}}(B \otimes C^{\mathrm{op}})$, $f_1 \in \mathrm{Hom}_{A \otimes B^{\mathrm{op}}}(K_1, K_1)$ and $f_2 \in \mathrm{Hom}_{B \otimes C^{\mathrm{op}}}(K_2, K_2)$. Then*

$$\mathrm{hh}_{A \otimes C^{\mathrm{op}}}(K_1 \overset{\mathrm{L}}{\underset{B}{\otimes}} K_2, f_1 \overset{\mathrm{L}}{\underset{B}{\otimes}} f_2) = \mathrm{hh}_{A \otimes B^{\mathrm{op}}}(K_1, f_1) \cup_B \mathrm{hh}_{B \otimes C^{\mathrm{op}}}(K_2, f_2).$$

Chapter 2

DG Affinity of DQ-modules

2.1 Introduction

Many classical results of complex algebraic or analytic geometry have a counterpart in the framework of Deformation Quantization modules (see [KS12]). Let us mention a few of them: Serre duality, convolution of coherent kernels, the construction of Hochschild classes for coherent DQ-modules in [KS12], a GAGA type theorem in [Che10] and Fourrier-Mukai transforms in [ABP11], etc.

In this chapter, extracted from [Pet11a], we give a non-commutative analogue of a famous result of Bondal-Van den Bergh asserting the dg affinity of quasi-compact quasi-separated schemes (see [BvdB03, Corollary 3.1.8]). In the framework of formal deformation algebroid stacks, the notion of quasi-coherent object is no more suited for this purpose. Thus, we introduce the notion of cohomologically complete and graded quasi-coherent objects (qcc for short). The qcc objects of the derived category $\mathsf{D}(\mathcal{A}_X)$, where $\mathcal{A}_X$ is a formal deformation algebroid stack, form a full triangulated subcategory of $\mathsf{D}(\mathcal{A}_X)$ denoted $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. This category can be thought of as the deformation of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ while deforming $\mathcal{O}_X$ into $\mathcal{A}_X$ (see Theorem 2.4.6). We prove that the image of a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is a compact generator of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. The existence of a compact generator in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is granted by a result of Bondal-Van den Bergh (see loc. cit.). Hence, the category $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is dg affine.

The study of generators in derived categories of geometric origin has been initiated by Beilinson in [Beĭ78]. The results of [BvdB03] have been refined by Rouquier in [Rou08] where he introduced a notion of dimension for triangulated categories. Recently, in [Toë] Toën generalized the results of Bondal and Van den Bergh and reinterpreted them in the framework of homotopical algebraic geometry.

This chapter is organised as follows. In the first part, we recall some classical material concerning generators in a triangulated category. We review, following [KS12], the notion of cohomological completeness and its link with the functor of $\hbar$-graduation. We finally state some results specific to deformation algebroid stacks on smooth algebraic varieties.

In the second part of the chapter, we introduce the triangulated category of qcc objects, that is to say objects of $\mathsf{D}(\mathcal{A}_X)$ that are cohomologically complete and whose associated graded module is quasi-coherent. We prove that the category $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ admits arbitrary coproducts. The coproduct is given by the cohomological completion of the usual direct sum (Proposition 2.3.11). Then we prove that $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is compactly generated (see Proposition 2.3.14 and Lemma 2.3.15). Relying on a theorem of Ravenel and Neeman (see [Rav84] and [Nee92]) we describe completely the compact objects of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ (see Theorem 2.3.19). They are objects of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ satisfying certain torsion conditions. Finally, we conclude this section by proving that $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is equivalent as a triangulated category to the derived category of a suitable dg algebra with bounded cohomology (see Theorem 2.3.21).

In the last section, we study qcc sheaves on an affine variety and prove that the equivalence of triangulated categories between $\mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X)$ and $\mathsf{D}^+(\mathcal{O}_X(X))$ lifts to an equivalence between $\mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X)$ and the triangulated category $\mathsf{D}^+_{cc}(\mathcal{A}_X(X))$ of cohomologically complete $\mathcal{A}_X(X)$-modules (see Theorem 2.4.6).

2.2 Review

2.2.1 Generators and compactness in triangulated categories: a review

We start with some classical definitions. See [Nee01], [BvdB03].

Recall that if $\mathcal{T}$ is a triangulated category, then a triangulated subcategory $\mathcal{B}$ of $\mathcal{T}$ is called thick if it is closed under isomorphisms and direct summands.

Definition 2.2.1. Let $\mathcal{S}$ be a set of objects of $\mathcal{T}$. The smallest thick triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$ is called the thick envelope of $\mathcal{S}$ and is denoted $\langle \mathcal{S} \rangle$. One says that $\mathcal{S}$ classically generates $\mathcal{T}$ if $\langle \mathcal{S} \rangle$ is equal to $\mathcal{T}$.

Definition 2.2.2. Let $\mathcal{T}$ be a triangulated category. Let $\mathfrak{G} = (G_i)_{i \in I}$ be a set of objects of $\mathcal{T}$. One says that $\mathfrak{G}$ generates $\mathcal{T}$ if for every $F \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(G_i[n], F) = 0$ for every $G_i \in \mathfrak{G}$ and $n \in \mathbb{Z}$, we have $F \simeq 0$.

Definition 2.2.3. Assume that $\mathcal{T}$ is cocomplete.

(a) An object L in $\mathcal{T}$ is compact if $\mathrm{Hom}_{\mathcal{T}}(L, \cdot)$ commutes with direct sums. We denote by $\mathcal{T}^c$ the full subcategory of $\mathcal{T}$ consisting of compact objects.

(b) The category $\mathcal{T}$ is compactly generated if it is generated by a set of compact objects.

The following result was proved independently by Ravenel and Neeman, see [Nee92] and [Rav84].

Theorem 2.2.4. *Assume that $\mathcal{T}$ is compactly generated. Then a set of objects $\mathcal{S} \subset \mathcal{T}^c$ classically generates $\mathcal{T}^c$ if and only if it generates $\mathcal{T}$.*

We give an inductive description of the thick envelope of a subset of a triangulated category. For that purpose, we introduce a multiplication on the set of full subcategories of a triangulated category. We follow closely the exposition of [BvdB03].

Definition 2.2.5. Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{C}$ and $\mathcal{D}$ be full subcategories of $\mathcal{T}$. One denotes by $\mathcal{C} \star \mathcal{D}$ the strictly full subcategory of $\mathcal{T}$ whose objects E occur in a triangle of the form

$$C \to E \to D \to C[1]$$

where $C \in \mathcal{C}$ and $B \in \mathcal{D}$.

Proposition 2.2.6. *The operation $\star$ is associative.*

Let $\mathcal{S}$ be a set of objects of $\mathcal{T}$. We denote by $\mathrm{add}(\mathcal{S})$ the smallest full subcategory in $\mathcal{T}$ which contains $\mathcal{S}$ and is closed under taking finite direct sums and shifts.
We denote by $\mathrm{smd}(\mathcal{S})$ the smallest full subcategory which contains $\mathcal{S}$ and is closed under taking direct summands.

Lemma 2.2.7. *If $\mathcal{C}$ and $\mathcal{D}$ are closed under finite direct sums, then* $\mathrm{smd}(\mathrm{smd}(\mathcal{C}) \star \mathcal{D}) = \mathrm{smd}(\mathcal{C} \star \mathcal{D})$.

Set

$$\langle \mathcal{S} \rangle_1 = \mathrm{smd}(\mathrm{add}(\mathcal{S})),$$

$$\langle \mathcal{S} \rangle_k = \mathrm{smd}(\underbrace{\langle \mathcal{S} \rangle_1 \star \ldots \star \langle \mathcal{S} \rangle_1}_{k\ factors}),$$

$$\langle \mathcal{S} \rangle = \bigcup_k \langle \mathcal{S} \rangle_k.$$

Then $\langle \mathcal{S} \rangle$ is the thick envelope of $\mathcal{S}$ (see Definition 2.2.1).

2.2.2 Recollections on algebraic categories

In this section, we recall some classical facts on algebraic categories, [Kel94], [Kel98b], [Kel06]. In this section R is a commutative unital ring.

Definition 2.2.8. A Frobenius category $\mathcal{E}$ is an exact category (in the sense of Quillen [Qui73]) with enough projective and injective objects such that an object is projective if and only if it is injective.

Let σ and $\sigma^{'}$ in $\mathcal{E}$. We denote by $\mathcal{N}(\sigma, \sigma^{'})$ the subgroup of $\mathrm{Hom}_{\mathcal{E}}(\sigma, \sigma^{'})$ formed by the maps that can be factorized through an injective object. We denote by $\underline{\mathcal{E}}$ the category with the same objects as $\mathcal{E}$ and whose morphisms spaces are the quotients $\mathrm{Hom}_{\mathcal{E}}(\sigma, \sigma^{'})/\mathcal{N}(\sigma, \sigma^{'})$. The category $\underline{\mathcal{E}}$ is called the stable category of $\mathcal{E}$. A classical result states that $\underline{\mathcal{E}}$ is a triangulated category.

Definition 2.2.9. One says that an R-linear triangulated category is algebraic if it is equivalent as a triangulated category to the stable category of an R-linear Frobenius category.

Proposition 2.2.10. *A triangulated subcategory of an algebraic triangulated category is algebraic.*

Proposition 2.2.11. *The derived category of an Abelian category is algebraic.*

We have the following theorem from [Kel98b] which is a consequence of [Kel94, Theorem 4.3]. If Λ is a dg category, we denote by $\mathsf{D}(\Lambda)$ its derived category in the sense of [Kel94] (note that $\mathsf{D}(\Lambda)$ is not a dg category).

Theorem 2.2.12. *Let $\mathcal{E}$ be a Frobenius category and set $\mathcal{T} = \underline{\mathcal{E}}$. Assume that $\mathcal{T}$ is cocomplete and has a compact generator G. Then, there is a dg algebra Λ and an equivalence of triangulated categories $F : \mathsf{D}(\Lambda) \to \mathcal{T}$ with $F(\Lambda) \xrightarrow{\sim} G$. In particular, we have*

$$\mathrm{H}^n(\Lambda) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\Lambda)}(\Lambda, \Lambda[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(G, G[n]), \ n \in \mathbb{Z}.$$

2.2.3 The case of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$

Let $(X, \mathcal{O}_X)$ be a scheme. We denote by $Qcoh(X)$ the category of quasi-coherent $\mathcal{O}_X$-modules. Its derived category is denoted by $\mathsf{D}(Qcoh(X))$. We write $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ for the full triangulated subcategory of $\mathsf{D}(\mathcal{O}_X)$ consisting of complexes with quasi-coherent cohomology.

Theorem 2.2.13 ([BN93])**.** *If X is a quasi-compact and separated scheme then the canonical functor $\mathsf{D}(Qcoh(X)) \to \mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is an equivalence.*

Definition 2.2.14. Let $(X, \mathcal{O}_X)$ be a scheme. A perfect complex on X is a complex of $\mathcal{O}_X$-modules which is locally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules of finite type. We denote by $\mathsf{D}_{\mathrm{perf}}(\mathcal{O}_X) \subset \mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ the category of perfect complexes.

In this chapter, we are interested in complex smooth algebraic varieties. We give a few properties of perfect complexes in this setting. Since X is an algebraic variety, X is a Noetherian topological space. Thus, a perfect complex in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is in $\mathsf{D}^b_{\mathrm{qcoh}}(\mathcal{O}_X)$. Since $\mathcal{O}_X$ is Noetherian it follows that $\mathsf{D}_{\mathrm{perf}}(\mathcal{O}_X) \subset \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$. Finally since X is smooth, we have $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X) \subset \mathsf{D}_{\mathrm{perf}}(\mathcal{O}_X)$. Thus, on a smooth algebraic variety, $\mathsf{D}_{\mathrm{perf}}(\mathcal{O}_X) = \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$.

Recall the following theorem from [BvdB03].

Theorem 2.2.15. *Assume that X is a quasi-compact and quasi-separated scheme. Then,*

(i) the compact objects in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ are the perfect complexes,

(ii) $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is generated by a single perfect complex.

As a corollary Bondal and Van den Bergh obtain

Theorem 2.2.16. *Assume that X is a quasi-compact quasi-separated scheme. Then $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is equivalent to $\mathsf{D}(\Lambda_0)$ for a suitable dg algebra Λ_0 with bounded cohomology.*

2.2.4 $\hbar$-graduation

The case of ringed space

In this section, X is a topological space and $\mathcal{R}$ is a $\mathbb{Z}[\hbar]_X$-algebra on X without $\hbar$-torsion. Throughout this text we assume that $\hbar$ is central in $\mathcal{R}$. We set $\mathcal{R}_0 = \mathcal{R}/\hbar\mathcal{R}$. We refer the reader to [KS12] for more details.

Definition 2.2.17. We denote by $\mathrm{gr}_\hbar : \mathsf{D}(\mathcal{R}) \to \mathsf{D}(\mathcal{R}_0)$ the left derived functor of the right exact functor $\mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(\mathcal{R}_0)$ given by $\mathcal{M} \mapsto \mathcal{M}/\hbar\mathcal{M}$. For $\mathcal{M} \in \mathsf{D}(\mathcal{R})$ we call $\mathrm{gr}_\hbar(\mathcal{M})$ the graded module associated to $\mathcal{M}$. We have

$$\mathrm{gr}_\hbar \mathcal{M} \simeq \mathcal{R}_0 \overset{\mathrm{L}}{\underset{\mathcal{R}}{\otimes}} \mathcal{M}.$$

Proposition 2.2.18. *(i) Let $\mathcal{K}_1 \in \mathsf{D}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{K}_2 \in \mathsf{D}(\mathcal{R})$. Then,*

$$\mathrm{gr}_\hbar(\mathcal{K}_1 \overset{\mathrm{L}}{\underset{\mathcal{R}}{\otimes}} \mathcal{K}_2) \simeq \mathrm{gr}_\hbar(\mathcal{K}_1) \overset{\mathrm{L}}{\underset{\mathcal{R}_0}{\otimes}} \mathrm{gr}_\hbar(\mathcal{K}_2).$$

(ii) Let $\mathcal{K}_i \in \mathsf{D}(\mathcal{R})$ $(i = 1, 2)$. Then

$$\mathrm{gr}_\hbar(\mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{K}_1, \mathcal{K}_2)) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathrm{gr}_\hbar \mathcal{K}_1, \mathrm{gr}_\hbar \mathcal{K}_2).$$

Proposition 2.2.19. *Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. The functor $\mathrm{gr}_\hbar : \mathsf{D}(\mathbb{Z}[\hbar]_Z) \to \mathsf{D}(\mathbb{Z}[\hbar]_Z)$, $Z = X, Y$ commutes with the operations of direct images $\mathrm{R}\, f_*$ and of inverse images.*

The case of algebroid stacks

We write $\mathbb{C}^\hbar$ for the ring $\mathbb{C}[[\hbar]]$. In this section $\mathcal{A}_X$ denotes a $\mathbb{C}^\hbar$-algebroid stack without $\hbar$-torsion. As in the previous subsection we refer the reader to [KS12].

Definition 2.2.20. Let $\mathcal{A}_X$ be a $\mathbb{C}^\hbar$-algebroid stack without $\hbar$-torsion on a topological space X. One denotes by $\mathrm{gr}_\hbar(\mathcal{A}_X)$ the $\mathbb{C}$-algebroid associated with the prestack $\mathfrak{S}$ given by:

$\mathrm{Ob}(\mathfrak{S}(U)) = \mathrm{Ob}(\mathcal{A}(U))$ for an open set U of X,
$\mathrm{Hom}_{\mathfrak{S}(U)}(\sigma, \sigma') = \mathrm{Hom}_{\mathcal{A}}(\sigma, \sigma')/\hbar \,\mathrm{Hom}_{\mathcal{A}}(\sigma, \sigma')$ for $\sigma, \sigma' \in \mathcal{A}(U)$.

There is a natural functor $\mathcal{A}_X \to \mathrm{gr}_\hbar(\mathcal{A}_X)$ of $\mathbb{C}$-algebroid stacks. This functor induces a functor

$$\iota_g : \mathrm{Mod}(\mathrm{gr}_\hbar \mathcal{A}_X) \to \mathrm{Mod}(\mathcal{A}_X).$$

The functor ι_g admits a left adjoint functor $\mathcal{M} \to \mathbb{C} \otimes_{\mathbb{C}^\hbar} \mathcal{M}$. The functor ι_g is exact and it induces a functor

$$\iota_g : \mathsf{D}(\mathrm{gr}_\hbar \mathcal{A}_X) \to \mathsf{D}(\mathcal{A}_X).$$

One extends the definition of $\mathrm{gr}_\hbar$ by

$$\mathrm{gr}_\hbar(\mathcal{M}) \simeq \mathrm{gr}_\hbar(\mathcal{A}_X) \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M} \simeq \mathbb{C} \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} \mathcal{M}.$$

The propositions of the preceding subsection concerning sheaves extend to the case of algebroid stacks. Finally we have the following important proposition.

Proposition 2.2.21. *The functor $\mathrm{gr}_\hbar$ and ι_g define pairs of adjoint functors $(\mathrm{gr}_\hbar, \iota_g)$ and $(\iota_g, \mathrm{gr}_\hbar[-1])$.*

Proof. We refer the reader to [KS12, Proposition 2.3.6]. □

2.2.5 Cohomologically Complete Modules

In this subsection, we briefly recall some facts about cohomologically complete modules. We closely follow [KS12] and refer the reader to it for an in depth treatment of the notion of cohomological completeness.

In this section, X is a topological space and $\mathcal{R}$ is a $\mathbb{Z}[\hbar]_X$-algebra without $\hbar$-torsion. We set $\mathcal{R}^{loc} := \mathbb{Z}[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}[\hbar]} \mathcal{R}$. The triangulated category $\mathsf{D}(\mathcal{R}^{loc})$ is equivalent to the full subcategory of $\mathsf{D}(\mathcal{R})$ consisting of objects $\mathcal{M}$ satisfying $\mathcal{M} \xrightarrow{\sim} \mathcal{R}^{loc} \overset{\mathrm{L}}{\underset{\mathcal{R}}{\otimes}} \mathcal{M}$ or equivalently $\mathrm{gr}_\hbar \mathcal{M} = 0$.

The right orthogonal category $\mathsf{D}(\mathcal{R}^{loc})^{\perp r}$ to the full subcategory $\mathsf{D}(\mathcal{R}^{loc})$ of $\mathsf{D}(\mathcal{R})$ is the full triangulated subcategory consisting of objects $\mathcal{M} \in \mathsf{D}(\mathcal{R})$ satisfying $\mathrm{Hom}_{\mathsf{D}(\mathcal{R})}(\mathcal{N}, \mathcal{M}) \simeq 0$ for any $\mathcal{N} \in \mathsf{D}(\mathcal{R}^{loc})$.

Definition 2.2.22. An object $\mathcal{M} \in \mathsf{D}(\mathcal{R})$ is cohomologically complete if it belongs to $\mathsf{D}(\mathcal{R}^{loc})^{\perp r}$. We write $\mathsf{D}_{\mathrm{cc}}(\mathcal{R})$ for $\mathsf{D}(\mathcal{R}^{loc})^{\perp r}$.

Propositions 2.2.23, 2.2.25 are proved in [KS12].

Proposition 2.2.23 ([KS12, Prop. 1.5.6])**.** *(i) For $\mathcal{M} \in \mathsf{D}(\mathcal{R})$, the following conditions are equivalent:*

(a) $\mathcal{M}$ is cohomologically complete,

(b) $\mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{loc}, \mathcal{M}) \simeq \mathrm{R}\mathcal{H}om_{\mathbb{Z}[\hbar]}(\mathbb{Z}[\hbar, \hbar^{-1}], \mathcal{M}) \simeq 0$,

(c) For any $x \in X$, $j = 0, 1$ and any $i \in \mathbb{Z}$,

$$\varinjlim_{x \in U} \mathrm{Ext}^j_{\mathcal{R}}(\mathcal{R}^{loc}, \mathrm{H}^i(U, \mathcal{M})) \simeq 0.$$

Here, U ranges over an open neighborhood system of x.

(ii) $\mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{loc}/\mathcal{R}, \mathcal{M})$ *is cohomologically complete for any $\mathcal{M} \in \mathsf{D}(\mathcal{R})$.*

(iii) For any $\mathcal{M} \in \mathsf{D}(\mathcal{R})$, there exists a distinguished triangle

$$\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1}$$

with $\mathcal{M}' \in \mathsf{D}(\mathcal{R}^{loc})$ and $\mathcal{M}'' \in \mathsf{D}_{\mathrm{cc}}(\mathcal{R})$.

(iv) Conversely, if

$$\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1}$$

is a distinguished triangle with $\mathcal{M}' \in \mathsf{D}(\mathcal{R}^{loc})$ and $\mathcal{M}'' \in \mathsf{D}_{\mathrm{cc}}(\mathcal{R})$, then $\mathcal{M}' \simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{loc}, \mathcal{M})$ *and* $\mathcal{M}'' \simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{loc}/\mathcal{R}[-1], \mathcal{M})$.

Lemma 2.2.24 ([KS12, Prop. 1.5.10])**.** *Assume that $\mathcal{M} \in \mathsf{D}(\mathcal{R})$ is cohomologically complete. Then* $\mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) \in \mathsf{D}(\mathbb{Z}_X[\hbar])$ *is cohomologically complete for any $\mathcal{N} \in \mathsf{D}(\mathcal{R})$.*

Proposition 2.2.25 ([KS12, Cor. 1.5.9])**.** *Let $\mathcal{M} \in \mathsf{D}(\mathcal{R})$ be a cohomologically complete object. If* $\mathrm{gr}_\hbar \mathcal{M} \simeq 0$, *then* $\mathcal{M} \simeq 0$.

Corollary 2.2.26. *Let $f : \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathsf{D}_{\mathrm{cc}}(\mathcal{R})$. If* $\mathrm{gr}_\hbar(f)$ *is an isomorphism then f is an isomorphism.*

The following result is implicit in [KS12]. We make it explicit since we use it frequently.

Proposition 2.2.27. *Let $\mathcal{M} \in \mathsf{D}(\mathcal{R})$ be such that there locally exists $n \in \mathbb{N}$ with $\hbar^n \mathcal{M} \simeq 0$. Then $\mathcal{M}$ is cohomologically complete.*

Proof. The action of $\hbar$ on $\mathcal{R}^{loc}$ is an isomorphism. Thus the morphism

$$\hbar\circ : \mathrm{R}\mathcal{H}om(\mathcal{R}^{loc}, \mathcal{M}) \to \mathrm{R}\mathcal{H}om(\mathcal{R}^{loc}, \mathcal{M})$$

is an isomorphism. The morphism

$$\circ\hbar : \mathrm{R}\mathcal{H}om(\mathcal{R}^{loc}, \mathcal{M}) \to \mathrm{R}\mathcal{H}om(\mathcal{R}^{loc}, \mathcal{M})$$

is locally nilpotent. Since $\hbar$ is central in $\mathcal{R}$, $\hbar\circ = \circ\hbar$. Hence, $\mathrm{R}\mathcal{H}om(\mathcal{R}^{loc}, \mathcal{M}) = 0$. $\square$

2.2.6 Modules over formal deformations after [KS12]

In this subsection, we recall some facts about formal deformation of ringed spaces. We refer the reader to [KS12] for DQ-modules. We refer to [Moe02], [KS06] for stacks and algebroid stacks. We denote by $\mathbb{C}^\hbar$ the ring $\mathbb{C}[[\hbar]]$.

Definition 2.2.28 ([KS12])**.** Let $(X, \mathcal{O}_X)$ be a commutative ringed space. Assume that $\mathcal{O}_X$ is a Noetherian sheaf of $\mathbb{C}$-algebras. A formal deformation algebra $\mathcal{A}_X$ of $\mathcal{O}_X$ is a sheaf of $\mathbb{C}^\hbar$-algebras such that

(i) $\hbar$ is central in $\mathcal{A}_X$

(ii) $\mathcal{A}_X$ has no $\hbar$-torsion

(iii) $\mathcal{A}_X$ is $\hbar$-complete

(iv) $\mathcal{A}_X/\hbar\mathcal{A}_X \simeq \mathcal{O}_X$ as sheaves of $\mathbb{C}$-algebras.

(v) There exists a base $\mathfrak{B}$ of open subsets of X such that for any $U \in \mathfrak{B}$ and any coherent $\mathcal{O}_X|_U$-module $\mathcal{F}$, we have $\mathrm{H}^n(U, F) = 0$ for any $n > 0$.

Remark 2.2.29. Clearly, on a complex algebraic variety, condition (v) of the preceding definition is satisfied.

Definition 2.2.30. A formal deformation algebroid stack $\mathcal{A}_X$ on X is a $\mathbb{C}^\hbar$-algebroid such that for each open set $U \subset X$ and each $\sigma \in \mathcal{A}_X(U)$, the $\mathbb{C}^\hbar$-algebra $\mathcal{E}nd_{\mathcal{A}_X}(\sigma)$ is a formal deformation algebra on U.

Remark 2.2.31. Note that formal deformation algebroid stack are called twisted deformations in [Yek09].

Let $\mathcal{A}_X$ be a formal deformation algebroid on X. We denote by $\mathfrak{Mod}(\mathbb{C}^\hbar_X)$ the $\mathbb{C}^\hbar$-linear stack of sheaves of $\mathbb{C}^\hbar$-modules over X and by $\mathrm{Mod}(\mathcal{A}_X)$ the category of functors $\mathrm{Fct}(\mathcal{A}_X, \mathfrak{Mod}(\mathbb{C}^\hbar_X))$. The category $\mathrm{Mod}(\mathcal{A}_X)$ is a Grothendieck category. For a module $\mathcal{M}$ over an algebroid $\mathcal{A}_X$ the local notions of being coherent, locally free etc. still make sense. We denote by $\mathsf{D}(\mathcal{A}_X)$ the derived category of $\mathrm{Mod}(\mathcal{A}_X)$, by $\mathsf{D}^b(\mathcal{A}_X)$ its bounded derived category and by $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ the full triangulated subcategory of $\mathsf{D}^b(\mathcal{A}_X)$ consisting of objects with coherent cohomologies.

Definition 2.2.32. We say that an algebroid is trivial if it is equivalent to the algebroid stack associated to a sheaf of rings.

From now on, we assume that X is a smooth algebraic variety endowed with the Zarisky topology. There are the following results (see Remark 2.1.17 of [KS12] due to Prof. Joseph Oesterlé).

Proposition 2.2.33. *On a smooth algebraic variety X, the group $\mathrm{H}^2(X, \mathcal{O}_X^\times)$ is zero.*

Corollary 2.2.34. *On a smooth algebraic variety, invertible $\mathcal{O}_X$-algebroid stacks are trivial.*

See [KS12, Definition 2.1.14 (ii)] or the appendix for a definition of invertible algebroid stack.

By the definition of the functor $\mathrm{gr}_\hbar$, it is clear that $\mathrm{gr}_\hbar \mathcal{A}_X$ is an $\mathcal{O}_X$ invertible algebroid and by Corollary 2.2.34 it follows that

$$\mathrm{gr}_\hbar \mathcal{A}_X \simeq \mathcal{O}_X. \tag{2.2.1}$$

Proposition 2.2.35. *The functor* $\mathrm{gr}_\hbar$ *induces a functor*

$$\mathrm{gr}_\hbar : \mathsf{D}(\mathcal{A}_X) \to \mathsf{D}(\mathcal{O}_X).$$

which sends bounded objects to bounded objects and objects with coherent cohomology to objects with coherents cohomology.

We have the following results from [KS12].

Proposition 2.2.36. *Let* $d \in \mathbb{N}$. *Assume that any coherent* $\mathcal{O}_X$*-module locally admits a resolution of length* $\leq d$ *by free* $\mathcal{O}_X$*-modules of finite rank. Let* $\mathcal{M}^\bullet$ *be a complex of* $\mathcal{A}_X$*-modules concentrated in degrees* $[a,b]$ *and assume that* $\mathrm{H}^i(\mathcal{M})$ *is coherent for all* i. *Then, in a neighborhood of each* $x \in X$, *there exists a quasi-isomorphism* $\mathcal{L}^\bullet \to \mathcal{M}^\bullet$ *where* $\mathcal{L}^\bullet$ *is a complex of free* $\mathcal{A}_X$*-modules of finite rank concentrated in degrees* $[a-d-1,b]$.

We have the following sufficient condition which is a corollary of more general results that ensure that under certain conditions, an algebroid stack of formal deformations is trivial (see [Kon01], [BGNT07], [CH11], [Yek09]).

Proposition 2.2.37. *Let* X *be a smooth algebraic variety endowed with a twisted deformation algebroid* $\mathcal{A}_X$ *in the sense of [Yek09]. If* $\mathrm{H}^1(X,\mathcal{O}_X) = \mathrm{H}^2(X,\mathcal{O}_X) = 0$, *then* $\mathcal{A}_X$ *is equivalent to the algebroid stack associated to a formal deformation algebra of* $\mathcal{O}_X$.

2.3 QCC Modules

From now on, we assume that X is a smooth algebraic variety endowed with the Zarisky topology and equipped with a formal deformation algebroid $\mathcal{A}_X$.

2.3.1 Graded quasi-coherent modules and quasi-coherent $\mathcal{O}_X$-modules

We introduce the category of graded quasi-coherent modules.

Definition 2.3.1. Let $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$. We say that $\mathcal{M}$ is graded quasi-coherent if $\mathrm{gr}_\hbar(\mathcal{M}) \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$. We denote by $\mathsf{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$ the full subcategory of $\mathsf{D}(\mathcal{A}_X)$ formed by graded quasi-coherent modules.

Proposition 2.3.2. *The category* $\mathsf{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$ *is a triangulated subcategory of* $\mathsf{D}(\mathcal{A}_X)$.

Proof. Obvious. □

2.3.2 QCC objects

In this subsection, we introduce the category of qcc-modules.

Definition 2.3.3. An object $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$ is qcc if it is graded quasi-coherent and cohomologically complete. The full subcategory of $\mathsf{D}(\mathcal{A}_X)$ formed by qcc-modules is denoted by $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.

Since, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X) = \mathsf{D}_{\mathrm{gqcoh}}(\mathcal{A}_X) \cap \mathsf{D}_{\mathrm{cc}}(\mathcal{A}_X)$, we have

Proposition 2.3.4. *The category* $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ *is a* $\mathbb{C}^\hbar$*-linear triangulated subcategory of* $\mathsf{D}(\mathcal{A}_X)$.

Proposition 2.3.5. *If $\mathcal{M} \in \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X)$ is such that $\mathrm{gr}_\hbar \mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$, then $\mathcal{M} \in \mathsf{D}^+_{\mathrm{coh}}(\mathcal{A}_X)$.*

Proof. It is a direct consequence of [KS12, Theorem 1.6.4]. □

Proposition 2.3.6. *If $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$, then $\mathcal{M} \in \mathsf{D}^b_{\mathrm{qcc}}(\mathcal{A}_X)$.*

Proof. It is a direct consequence of [KS12, Theorem 1.6.1]. □

We now prove that $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is cocomplete. For that, we first prove that $\mathsf{D}_{cc}(\mathcal{A}_X)$ is cocomplete.

Definition 2.3.7. We denote by $(\cdot)^{\mathrm{cc}}$ the functor

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_X}((\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1], \cdot) : \mathsf{D}(\mathcal{A}_X) \to \mathsf{D}(\mathcal{A}_X).$$

We call this functor the functor of cohomological completion.

By Proposition 2.2.23, the functor of cohomological completion takes its values in $\mathsf{D}_{cc}(\mathcal{A}_X)$.

The following exact sequence

$$0 \to \mathcal{A}_X \to \mathcal{A}_X^{loc} \to \mathcal{A}_X^{loc}/\mathcal{A}_X \to 0. \tag{2.3.1}$$

induces a morphism

$$\mathcal{A}_X^{loc}/\mathcal{A}_X[-1] \to \mathcal{A}_X.$$

This morphism yields a morphism of functors

$$cc : \mathrm{id} \to (\cdot)^{cc}. \tag{2.3.2}$$

Proposition 2.3.8. *The morphism of functors*

$$\mathrm{gr}_\hbar(cc) : \mathrm{gr}_\hbar \circ \mathrm{id} \to \mathrm{gr}_\hbar \circ (\cdot)^{cc}$$

is an isomorphism.

Proof. Let $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$. Applying Lemma 2.2.18 (ii), we get the following isomorphism

$$\begin{aligned}\mathrm{gr}_\hbar(\mathcal{M}^{cc}) &\simeq \mathrm{gr}_\hbar \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_X}((\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1], \mathcal{M}) \\ &\simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathrm{gr}_\hbar \mathcal{A}_X}(\mathrm{gr}_\hbar(\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1], \mathrm{gr}_\hbar \mathcal{M}).\end{aligned}$$

Applying the functor $\mathrm{gr}_\hbar$ to (2.3.1) and noticing that $\mathrm{gr}_\hbar \mathcal{A}_X^{loc} \simeq 0$, we deduce that $\mathrm{gr}_\hbar(\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1] \simeq \mathrm{gr}_\hbar \mathcal{A}_X$. Hence, $\mathrm{gr}_\hbar(\mathcal{M}^{cc}) \simeq \mathrm{gr}_\hbar \mathcal{M}$. □

Definition 2.3.9. Let $(\mathcal{M}_i)_{i \in I}$ be a family of objects of $\mathsf{D}_{\mathrm{cc}}(\mathcal{A}_X)$. We set

$$\overline{\bigoplus_{i \in I}} \mathcal{M}_i = \left(\bigoplus_{i \in I} \mathcal{M}_i \right)^{\mathrm{cc}}$$

where $\bigoplus$ denotes the direct sum in the category $\mathsf{D}(\mathcal{A}_X)$.

Proposition 2.3.10. *The category $\mathsf{D}_{\mathrm{cc}}(\mathcal{A}_X)$ admits direct sums. The direct sum of the family $(\mathcal{M}_i)_{i \in I}$ is given by $\overline{\bigoplus_{i \in I}} \mathcal{M}_i$.*

Proof. Let $(\mathcal{M}_i)_{i\in I}$ be a family of elements of $\mathsf{D}_{\mathrm{cc}}(\mathcal{A}_X)$. By Proposition 2.2.23 (*ii*), $\overline{\bigoplus_{i\in I}}\mathcal{M}_i$ is cohomologically complete.

Using the natural transformation (2.3.2) we obtain a morphism

$$cc : \bigoplus_{i\in I} \mathcal{M}_i \to \overline{\bigoplus_{i\in I}}\mathcal{M}_i.$$

It remains to shows that for all $\mathcal{F} \in \mathsf{D}_{\mathrm{cc}}(\mathcal{A}_X)$, the morphism (2.3.2) induces an isomorphism

$$\mathrm{Hom}_{\mathcal{A}_X}(\overline{\bigoplus_{i\in I}}\mathcal{M}_i, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_X}(\bigoplus_{i\in I} \mathcal{M}_i, \mathcal{F}). \tag{2.3.3}$$

It is enough to prove the isomorphism

$$\mathrm{R}\mathcal{H}om_{\mathcal{A}_X}(\overline{\bigoplus_{i\in I}}\mathcal{M}_i, \mathcal{F}) \to \mathrm{R}\mathcal{H}om_{\mathcal{A}_X}(\bigoplus_{i\in I} \mathcal{M}_i, \mathcal{F}). \tag{2.3.4}$$

Since both terms of (2.3.4) are cohomologically complete by Lemma 2.2.24, it remains to check the isomorphism on the associated graded map. Applying $\mathrm{gr}_\hbar$ to (2.3.4) and using Lemma 2.2.18 (*ii*) and Proposition 2.3.8 , we obtain an isomorphism

$$\mathrm{R}\mathcal{H}om_{\mathrm{gr}_\hbar \mathcal{A}_X}(\mathrm{gr}_\hbar(\overline{\bigoplus_{i\in I}}\mathcal{M}_i), \mathrm{gr}_\hbar \mathcal{F}) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_{\mathrm{gr}_\hbar \mathcal{A}_X}(\mathrm{gr}_\hbar(\bigoplus_{i\in I} \mathcal{M}_i), \mathrm{gr}_\hbar \mathcal{F}).$$

which proves the isomorphism (2.3.4). □

Proposition 2.3.11. *The category* $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ *admits direct sums. The direct sum of the family* $(\mathcal{M}_i)_{i\in I}$ *is given by* $\overline{\bigoplus_{i\in I}}\mathcal{M}_i$.

Proof. We know by Proposition 2.3.10, that $\mathsf{D}_{\mathrm{cc}}(\mathcal{A}_X)$ admits direct sums and it is given by $\overline{\bigoplus}$. Let $(\mathcal{M}_i)_{i\in I} \in \mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. Then, by Proposition 2.3.8, $\mathrm{gr}_\hbar \overline{\bigoplus_{i\in I}}\mathcal{M}_i = \bigoplus_{i\in I} \mathrm{gr}_\hbar \mathcal{M}_i$. It follows that $\overline{\bigoplus_{i\in I}}\mathcal{M}_i \in \mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. □

2.3.3 Compact objects and generators in $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$

In this subsection, we show that $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is generated by a compact generator and we describe its compact objects. We start by proving some additional properties on the functors $\mathrm{gr}_\hbar$ and ι_g which are defined in subsection 2.2.4.

Concerning ι_g, recall that there is a functor of stacks $\mathcal{A}_X \to \mathrm{gr}_\hbar(\mathcal{A}_X) \simeq \mathcal{O}_X$ inducing

$$\iota_g : \mathsf{D}(\mathcal{O}_X) \to \mathsf{D}(\mathcal{A}_X)$$

Notice that $\mathcal{O}_X$ can be endowed with a structure of left $\mathcal{O}_X$-module and right $\mathcal{A}_X$-module. When endowed with such structures we denote it by $\mathcal{O}_{XA}$. The module $\mathcal{O}_{XA}$ belongs to $\mathsf{D}^b(\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{A}_X^{\mathrm{op}})$. Similarly we have ${}_A\mathcal{O}_X \in \mathsf{D}^b(\mathcal{A}_X \otimes_{\mathbb{C}} \mathcal{O}_X^{\mathrm{op}})$. When $\mathcal{O}_X$ is endowed with its structure of $\mathcal{A}_X \otimes \mathcal{A}_X^{\mathrm{op}}$-module we denote it by ${}_A\mathcal{O}_{XA} \in \mathsf{D}^b(\mathcal{A}_X \otimes_C \mathcal{A}_X^{\mathrm{op}})$. With these notation, we have for $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$ and $\mathcal{N} \in \mathsf{D}(\mathcal{O}_X)$

$$\mathrm{gr}_\hbar(\mathcal{M}) = \mathcal{O}_{XA} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M} \qquad \text{and} \qquad \begin{aligned}\iota_g(\mathcal{N}) &= {}_A\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{N} \\ &= {}_A\mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{N}.\end{aligned}$$

Hence

$$\begin{aligned}\iota_g \circ \mathrm{gr}_\hbar(\mathcal{M}) &= {}_A\mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{O}_{XA} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}\\ &\simeq {}_A\mathcal{O}_{XA} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}.\end{aligned}$$

Proposition 2.3.12. *For every $\mathcal{N} \in \mathsf{D}(\mathcal{O}_X)$,*

$$\iota_g \circ \mathrm{gr}_\hbar \circ \iota_g(\mathcal{N}) \simeq \iota_g(\mathcal{N}) \oplus \iota_g(\mathcal{N})[1]$$

Proof. We have the exact sequence of $\mathcal{A}_X \otimes \mathcal{A}_X^{\mathrm{op}}$-modules

$$0 \longrightarrow \mathcal{A}_X \xrightarrow{\hbar} \mathcal{A}_X \longrightarrow {}_A\mathcal{O}_{XA} \longrightarrow 0.$$

Thus, for every $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$, we have $\iota_g \circ \mathrm{gr}_\hbar(\mathcal{M}) \simeq (\mathcal{A}_X[1] \xrightarrow{\hbar} \mathcal{A}_X) \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}$. Hence, for $\mathcal{N} \in \mathsf{D}(\mathcal{O}_X)$, $\iota_g \circ \mathrm{gr}_\hbar \circ \iota_g(\mathcal{N}) \simeq \iota_g(\mathcal{N}) \oplus \iota_g(\mathcal{N})[1]$. □

Corollary 2.3.13. *If $\mathcal{N} \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$, then $\iota_g(\mathcal{N}) \in \mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.*

Proof. Let $\mathcal{N}$ in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ and consider $\mathrm{gr}_\hbar \circ \iota_g(\mathcal{N})$. We compute $\mathrm{H}^i(\mathrm{gr}_\hbar \circ \iota_g(\mathcal{N}))$.

$$\begin{aligned}\iota_g(\mathrm{H}^i(\mathrm{gr}_\hbar \circ \iota_g(\mathcal{N}))) &\simeq \mathrm{H}^i(\iota_g \circ \mathrm{gr}_\hbar \circ \iota_g(\mathcal{N})) && \text{(exactness of } \iota_g)\\ &\simeq \mathrm{H}^i(\iota_g(\mathcal{N}) \oplus \iota_g(\mathcal{N})[1]) && \text{(Proposition 2.3.12)}\\ &\simeq \iota_g(\mathrm{H}^i(\mathcal{N}) \oplus \mathrm{H}^{i+1}(\mathcal{N})) && \text{(exactness of } \iota_g).\end{aligned}$$

The functor $\iota_g : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{A}_X)$ is fully faithful. Thus,

$$\mathrm{H}^i(\mathrm{gr}_\hbar \circ \iota_g(\mathcal{N})) \simeq \mathrm{H}^i(\mathcal{N}) \oplus \mathrm{H}^{i+1}(\mathcal{N}).$$

It follows that $\iota_g(\mathcal{N})$ is in $\mathsf{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$ and it is cohomologicaly complete by Proposition 2.2.27. □

Proposition 2.3.14. *If $\mathcal{G}$ is a generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$, then $\iota_g(\mathcal{G})$ is a generator of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$*

Proof. By Corollary 2.3.13, $\iota_g(\mathcal{G})$ is in $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.

Let $\mathcal{M} \in \mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ with $\mathrm{RHom}_{\mathcal{A}_X}(\iota_g(\mathcal{G}), \mathcal{M}) = 0$. By Proposition 2.2.21, we have

$$\mathrm{RHom}_{\mathcal{A}_X}(\iota_g(\mathcal{G}), \mathcal{M}) \simeq \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{G}, \mathrm{gr}_\hbar(\mathcal{M})[-1]).$$

Thus, $\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{G}, \mathrm{gr}_\hbar(\mathcal{M})[-1]) \simeq 0$ and $\mathrm{gr}_\hbar(\mathcal{M})[-1]$ is in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ thus $\mathrm{gr}_\hbar(\mathcal{M})[-1] \simeq 0$. Since $\mathcal{M}$ is cohomologically complete, $\mathcal{M} \simeq 0$. □

Lemma 2.3.15. *If $\mathcal{F} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ satisfies $\mathcal{A}_X^{loc} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{F} = 0$ then $\mathcal{F}$ is compact in $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.*

Proof. Let $(\mathcal{M}_i)_{i \in I}$ be a family of objects of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. By the adjunction between $(\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1] \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \cdot$ and $\mathrm{R}\mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1], \cdot)$, we have

$$\mathrm{Hom}_{\mathcal{A}_X}(\mathcal{F}, \overline{\bigoplus_{i \in I}} \mathcal{M}_i) \simeq \mathrm{Hom}_{\mathcal{A}_X}((\mathcal{A}_X^{loc}/\mathcal{A}_X)[-1] \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{F}, \bigoplus_{i \in I} \mathcal{M}_i).$$

In $\mathrm{Mod}(\mathcal{A}_X \otimes \mathcal{A}_X^{\mathrm{op}})$, we have the exact sequence (2.3.1). Tensoring by $\mathcal{F}$, we obtain that $\mathcal{A}_X^{loc}/\mathcal{A}_X[-1] \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{F}$ is isomorphic to $\mathcal{F}$. It follows that

$$\mathrm{Hom}_{\mathcal{A}_X}(\mathcal{F}, \overline{\bigoplus_{i\in I}} \mathcal{M}_i) \simeq \mathrm{Hom}_{\mathcal{A}_X}(\mathcal{F}, \bigoplus_{i\in I} \mathcal{M}_i). \tag{2.3.5}$$

The module $\mathcal{F}$ belongs to $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$. Using Proposition 2.2.36, and the fact that X is a Noetherian topological space of finite dimension, we have by [LH09, Corollary 3.9.3.2] [1]

$$\mathrm{Hom}_{\mathcal{A}_X}(\mathcal{F}, \bigoplus_{i\in I} \mathcal{M}_i) \simeq \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{A}_X}(\mathcal{F}, \mathcal{M}_i) \tag{2.3.6}$$

which together with (2.3.5) proves the lemma. □

Corollary 2.3.16. *If $\mathcal{F}$ is compact in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ then $\iota_g(\mathcal{F})$ is compact in $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.*

Proof. Obviously, $\mathcal{A}_X^{loc} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \iota_g(\mathcal{F}) = 0$. Since $\mathcal{F}$ is compact in $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$, it has coherent cohomology. Thus, its image by ι_g belongs to $\mathsf{D}_{\mathrm{coh}}(\mathcal{A}_X)$. The result follows from Lemma 2.3.15. □

Corollary 2.3.17. *If $\mathcal{G}$ is a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ then $\iota_g(\mathcal{G})$ is a compact generator of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. In particular, the category $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is compactly generated.*

Proof. By Theorem 2.2.15 due to Bondal and Van den Bergh, $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ has a compact generator. Then, the second claim of the corollary is a direct consequence of the first one. □

Lemma 2.3.18. *Let X be a smooth complex algebraic variety endowed with a DQ-algebroid.*

(i) If $\mathcal{G}$ is a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ then $\mathrm{gr}_\hbar\, \iota_g\mathcal{G}$ is still a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$.

(ii) One has $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X) = \langle \mathrm{gr}_\hbar\, \iota_g(\mathcal{G})\rangle$.

Proof. (i) Let $\mathcal{M} \in \mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ and assume that $\mathrm{RHom}_{\mathcal{O}_X}(\mathrm{gr}_\hbar\, \iota_g\mathcal{G}, \mathcal{M}) \simeq 0$. We have

$$\begin{aligned}\mathrm{RHom}_{\mathcal{O}_X}(\mathrm{gr}_\hbar\, \iota_g\mathcal{G}, \mathcal{M}) &\simeq \mathrm{RHom}_{\mathcal{A}_X}(\iota_g\mathcal{G}, \iota_g\mathcal{M}) \\ &\simeq 0.\end{aligned}$$

By Corollary 2.3.17 it follows that $\iota_g(\mathcal{M}) \simeq 0$. Hence, for every $i \in \mathbb{Z}$, $\mathrm{H}^i(\iota_g\mathcal{M}) \simeq 0$. Since $\iota_g : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{A}_X)$ is fully faithful and exact, we have $\mathrm{H}^i(\mathcal{M}) \simeq 0$. It follows that $\mathcal{M} \simeq 0$. Moreover, $\mathrm{gr}_\hbar\, \iota_g\mathcal{G}$ is coherent which proves the claim.

(ii) On a complex smooth algebraic variety the category of compact objects is $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_X)$. Hence the results follows from Theorem 2.2.4.

□

Theorem 2.3.19. *An object $\mathcal{M}$ of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is compact if and only if $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ and $\mathcal{A}_X^{loc} \otimes_{\mathcal{A}_X} \mathcal{M} = 0$.*

1. In [LH09] the results is stated for a concentraded scheme map. On a Noetherian topological space X of finite dimension, $\mathrm{R}\Gamma(X, \cdot)$ is of finite cohomological dimension on $\mathrm{Mod}(\mathbb{Z}_X)$. Thus the proof of [LH09, Corollary 3.9.3.2] goes without any changes for abelian sheaves on X .

Proof. The condition is sufficient by Lemma 2.3.15. Let $\mathcal{G}$ be a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$. By Theorem 2.2.4, we know that the set of compact objects of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is equivalent to the thick envelope $\langle \iota_g(\mathcal{G})\rangle$ of $\iota_g(\mathcal{G})$. Let us show that if $\mathcal{F} \in \langle \iota_g(\mathcal{G})\rangle$ then $\mathcal{F} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ and $\mathcal{A}_X^{loc} \otimes_{\mathcal{A}_X} \mathcal{F} = 0$.

Let $\mathcal{K}er$ be the full subcategory of $\mathsf{D}_{qcc}(\mathcal{A}_X)$ such that

$$\mathrm{Ob}(\mathcal{K}er) = \{\mathcal{F} \in \mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X) \mid \mathcal{A}_X^{loc} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{F} \simeq 0\}.$$

By Proposition 1.3.3, $\mathcal{K}er$ is a thick subcategory of $\mathsf{D}_{qcc}(\mathcal{A}_X)$. Moreover, the category $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ is also a thick subcategory of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. Then, the category $\mathcal{K}o = \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X) \cap \mathcal{K}er$ is a thick subcategory of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ containing $\iota_g(\mathcal{G})$. Since $\langle \iota_g(\mathcal{G})\rangle$ is the smallest thick subcategory of $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ containing $\iota_g(\mathcal{G})$, it follows that $\langle \iota_g(\mathcal{G})\rangle \subset \mathcal{K}o$ which proves the claim. $\square$

Remark 2.3.20. In the case of the ring $\mathbb{C}^\hbar$, Theorem 2.3.19 implies that $\mathbb{C}^\hbar$ is not a compact object of $\mathsf{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$. This can be checked directly as follow. Notice that

$$\mathrm{Hom}_{\mathbb{C}^\hbar}(\mathbb{C}^\hbar, \overline{\bigoplus_{n\in\mathbb{Z}}}\mathbb{C}^\hbar) \simeq \mathrm{Ext}^1_{\mathbb{C}^\hbar}(\mathbb{C}^{\hbar,loc}/\mathbb{C}^\hbar, \bigoplus_{n\in\mathbb{Z}}\mathbb{C}^\hbar).$$

Then by [KS12, §1.5] it follows that $\mathrm{Ext}^1_{\mathbb{C}^\hbar}(\mathbb{C}^{\hbar,loc}/\mathbb{C}^\hbar, \bigoplus_{n\in\mathbb{Z}}\mathbb{C}^\hbar)$ is isomorphic to the $\hbar$-adic completion of $\bigoplus_{n\in\mathbb{Z}}\mathbb{C}^\hbar$. Now, assume that $\mathbb{C}^\hbar$ is compact in $\mathsf{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$. Then

$$\mathrm{Hom}_{\mathbb{C}^\hbar}(\mathbb{C}^\hbar, \overline{\bigoplus_{n\in\mathbb{Z}}}\mathbb{C}^\hbar) \simeq \bigoplus_{n\in\mathbb{Z}}\mathbb{C}^\hbar.$$

But $\bigoplus_{n\in\mathbb{Z}}\mathbb{C}^\hbar$ is not $\hbar$-adically complete. Thus, $\mathbb{C}^\hbar$ is not compact in $\mathsf{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$.

2.3.4 DG Affinity of DQ-modules

In this subsection, we prove that the category of qcc DQ-modules is DG affine.

Theorem 2.3.21. *Assume X is a smooth complex algebraic variety endowed with a deformation algebroid $\mathcal{A}_X$. Then, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is equivalent to $\mathsf{D}(\Lambda)$ for a suitable dg algebra Λ with bounded cohomology.*

Proof. By Proposition 2.3.4, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is a $\mathbb{C}^\hbar$-linear triangulated subcategory of $\mathsf{D}(\mathcal{A}_X)$ which is algebraic by Proposition 2.2.11. It follows, by Proposition 2.2.10, that $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is algebraic. By Proposition 2.3.11, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is a cocomplete category. Moreover, by Corollary 2.3.17, $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ has a compact generator. It follows from Theorem 2.2.12 that $\mathsf{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is equivalent to the derived category of a dg algebra Λ such that

$$\mathrm{H}^n(\Lambda) \simeq \mathrm{Hom}_{\mathcal{A}_X}(\iota_g(\mathcal{G}), \iota_g(\mathcal{G})[n]),\ n \in \mathbb{Z}.$$

Using the adjunction between ι_g and $\mathrm{gr}_\hbar[-1]$ and [BvdB03, Lemma 3.3.8], we get that the cohomology of Λ is bounded. $\square$

Example 2.3.22. Let us compute such a dg algebra Λ in the case of $X = \mathrm{pt}$. In this setting, $\mathcal{A}_X = \mathbb{C}^\hbar$ and we get an equivalence $\mathsf{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \simeq \mathsf{D}(\Lambda)$. The $\mathbb{C}^\hbar$-module $\mathbb{C}$ is a compact generator of the category $\mathsf{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$. The complex $\mathscr{F}^\bullet := 0 \to \mathbb{C}^\hbar \xrightarrow{\hbar} \mathbb{C}^\hbar \to 0$ is a free resolution of $\mathbb{C}$ as a $\mathbb{C}^\hbar$-module. Then, we can chose

$$\Lambda \simeq \mathrm{RHom}_{\mathbb{C}^\hbar}(\mathscr{F}^\bullet, \mathscr{F}^\bullet).$$

It follows that

$$\Lambda^{-1} \simeq \mathbb{C}^\hbar \qquad \Lambda^0 \simeq \mathbb{C}^\hbar \oplus \mathbb{C}^\hbar \qquad \Lambda^1 \simeq \mathbb{C}^\hbar$$

and that

$$d^{-1} = \begin{pmatrix} \hbar \\ \hbar \end{pmatrix} \qquad d^0 = \begin{pmatrix} \hbar & -\hbar \end{pmatrix}.$$

2.4 QCC Sheaves on Affine Varieties

We assume that X is a smooth algebraic affine variety and that $\mathcal{A}_X$ is a sheaf of formal deformations (see Proposition 2.2.37). We set $A = \Gamma(X, \mathcal{A}_X)$, $B = \Gamma(X, \mathcal{O}_X)$ and $a_X : X \to \{pt\}$. As usual we denote by A_X (resp. B_X) the constant sheaf with stalk A (resp. B).

2.4.1 Preliminary results

Lemma 2.4.1. *The A_X-module $\mathcal{A}_X$ is flat.*

Proof. It is a direct consequence of [KS12, Theorem 1.6.6]. □

Lemma 2.4.2. *Let $f : X \to Y$ be a morphism of varieties and let $\mathcal{M} \in \mathsf{D}(f^{-1}\mathcal{A}_Y)$. Then we have*

$$\mathrm{R}\, f_* \mathcal{M}^{cc} \simeq (\mathrm{R}\, f_* \mathcal{M})^{\mathrm{cc}} \text{ in } \mathsf{D}(\mathcal{A}_Y).$$

Proof. It is a slight modification of [KS12, 1.5.12] □

We recall the following classical result.

Lemma 2.4.3. *Let $M \in \mathsf{D}(B)$. The canonical morphism*

$$M \to \mathrm{R}\Gamma(X, \mathcal{O}_X \otimes_{B_X} a_X^{-1} M) \tag{2.4.1}$$

is an isomorphism.

Proof. If M is concentrated in degree zero, the result follows directly from the equivalence of categories between $Qcoh(\mathcal{O}_X)$ and $\mathrm{Mod}(B)$. The result extends immediately to the derived category because $\mathcal{O}_X \otimes_{B_X} \cdot$ is an exact functor and because $\Gamma(X, \cdot)$ is exact on $Qcoh(\mathcal{O}_X)$ since X is affine. □

The functors $for^{(\cdot)}_{(\cdot)}$, below, are forgetful functors. We set

$$\begin{aligned} \mathrm{gr}_\hbar^{B_X} : \mathsf{D}(A_X) &\to \mathsf{D}(B_X) & \mathrm{gr}_\hbar^{\mathbb{C}_X} : \mathsf{D}(\mathbb{C}^\hbar_X) &\to \mathsf{D}(\mathbb{C}_X) & \mathrm{gr}_\hbar : \mathsf{D}(\mathcal{A}_X) &\to \mathsf{D}(\mathcal{O}_X) \\ \mathcal{M} &\mapsto B_X \overset{\mathrm{L}}{\underset{A_X}{\otimes}} \mathcal{M} & \mathcal{M} &\mapsto \mathbb{C}_X \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar_X}{\otimes}} \mathcal{M} & \mathcal{M} &\mapsto \mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}. \end{aligned}$$

Proposition 2.4.4. *The diagrams below are quasi-commutative.*

$$
(i)\quad
\begin{tikzcd}
\mathsf{D}(A_X) \arrow[r, "for_{A_X}"] & \mathsf{D}(\mathbb{C}^\hbar_X) \arrow[d, equal] \\
\mathsf{D}(\mathcal{A}_X) \arrow[u, "for^{A_X}_{\mathcal{A}_X}"] \arrow[r, "for_{\mathcal{A}_X}"] \arrow[d, "\mathrm{gr}_\hbar"] & \mathsf{D}(\mathbb{C}^\hbar_X) \arrow[d, "\mathrm{gr}_\hbar^{\mathbb{C}_X}"] \\
\mathsf{D}(\mathcal{O}_X) \arrow[r, "for_{\mathcal{O}_X}"] \arrow[d, "for^{B_X}_{\mathcal{O}_X}"] & \mathsf{D}(\mathbb{C}_X) \arrow[d, equal] \\
\mathsf{D}(B_X) \arrow[r, "for_{B_X}"] & \mathsf{D}(\mathbb{C}_X)
\end{tikzcd}
\qquad
(ii)\quad
\begin{tikzcd}
\mathsf{D}(A_X) \arrow[r, "for_{A_X}"] \arrow[d, "\mathrm{gr}_\hbar^{B_X}"] & \mathsf{D}(\mathbb{C}^\hbar_X) \arrow[d, "\mathrm{gr}_\hbar^{\mathbb{C}_X}"] \\
\mathsf{D}(B_X) \arrow[r, "for_{B_X}"] & \mathsf{D}(\mathbb{C}_X).
\end{tikzcd}
$$

Proof. (i) We start by proving that $for_{\mathcal{O}_X} \circ \mathrm{gr}_\hbar = \mathrm{gr}_\hbar^{\mathbb{C}_X} \circ for_{\mathcal{A}_X}$. Let $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$. We have

$$
\begin{aligned}
for_{\mathcal{O}_X} \circ \mathrm{gr}_\hbar(\mathcal{M}) &= \mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M} \\
&\simeq \mathbb{C} \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} \mathcal{A}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M} \\
&\simeq \mathbb{C} \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} for_{\mathcal{A}_X}(\mathcal{M}) \\
&\simeq \mathrm{gr}_\hbar^{\mathbb{C}} \circ for_{\mathcal{A}_X}(\mathcal{M}).
\end{aligned}
$$

The other commutation relations are similarly proved and are left to the reader.

(ii) Similar to (i).

□

Proposition 2.4.5. *Let* $\mathcal{M} \in \mathsf{D}^+(\mathcal{A}_X)$. *There is an isomorphism in* $\mathsf{D}^+(B)$

$$
B \overset{\mathrm{L}}{\underset{A}{\otimes}} \mathrm{R}\Gamma(X, \mathcal{M}) \simeq \mathrm{R}\Gamma(X, \mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}).
$$

Proof. Let $\mathcal{M}$ be an object of $\mathsf{D}^+(\mathcal{A}_X)$. The morphism of rings $B_X \to \mathcal{O}_X$ gives us a morphism in $\mathsf{D}^+(B_X)$

$$
\alpha : B_X \overset{\mathrm{L}}{\underset{A_X}{\otimes}} \mathcal{M} \to \mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M} \tag{2.4.2}
$$

and by Proposition 2.4.4, $for_{B_X}(\alpha) = \mathrm{id}_{\mathbb{C} \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} \mathcal{M}}$. Since for_{B_X} is conservative, it follows that α is an isomorphism in $\mathsf{D}^+(B_X)$.

For the sake of brevity, we write a_{X*} instead of $\mathrm{R}\, a_{X*}$ The adjunction between a_X^{-1} and a_{X*} gives a morphism in $\mathsf{D}^+(B)$

$$
pr : B \overset{\mathrm{L}}{\underset{A}{\otimes}} a_{X*}(\mathcal{M}) \to a_{X*}(a_X^{-1} B \overset{\mathrm{L}}{\underset{A_X}{\otimes}} \mathcal{M}). \tag{2.4.3}
$$

Applying for_B to the morphism pr, we obtain

$$\mathbb{C} \overset{\mathrm{L}}{\underset{\mathbb{C}^{\hbar}}{\otimes}} \mathrm{R}\, a_{X*}\mathcal{M} \to \mathrm{R}\, a_{X*}(\mathbb{C}_X \overset{\mathrm{L}}{\underset{\mathbb{C}^{\hbar}_X}{\otimes}} \mathcal{M}). \tag{2.4.4}$$

In the category $\mathsf{D}^+(\mathbb{C}^{\hbar}_X)$, $\mathbb{C}_X$ admits a free resolution given by $\mathbb{C}^{\hbar}_X \xrightarrow{\hbar} \mathbb{C}^{\hbar}_X$. Hence, by the projection formula, we get that the morphism (2.4.4) is an isomorphism in $\mathsf{D}(\mathbb{C}_X)$. Since for_B is conservative, pr is an isomorphism. The composition $a_{X*}(\alpha) \circ pr$ gives us the desired isomorphism.

□

2.4.2 QCC sheaves on affine varieties

We define the two functors:

$$\Phi : \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathsf{D}^+_{\mathrm{cc}}(A),\ \Phi(\mathcal{M}) = \mathrm{R}\Gamma(X, \mathcal{M})$$

and

$$\Psi : \mathsf{D}^+_{\mathrm{cc}}(A) \to \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X), \Psi(M) = (\mathcal{A}_X \otimes_{A_X} a_X^{-1} M)^{\mathrm{cc}}.$$

Theorem 2.4.6. *Let X be a smooth affine variety. The functors Φ and Ψ are equivalences of triangulated categories, are inverses one to each other and the diagram below is quasi-commutative*

$$\begin{tikzcd}
\mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X) \arrow[r, shift left, "\Phi"] \arrow[d, "\mathrm{gr}_{\hbar}"] & \mathsf{D}^+_{\mathrm{cc}}(A) \arrow[l, shift left, "\Psi"] \arrow[d, "\mathrm{gr}_{\hbar}"] \\
\mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X) \arrow[r, shift left, "\mathrm{R}\Gamma(X,\cdot)"] & \mathsf{D}^+(B). \arrow[l, shift left, "\mathcal{O}_X \otimes_{B_X} \cdot"]
\end{tikzcd}$$

Proof. Let $\mathcal{M} \in \mathsf{D}^+_{\mathrm{qcc}}(\mathcal{A}_X)$. By definition,

$$\Psi \circ \Phi(\mathcal{M}) = \mathrm{R}\mathcal{H}om_{\mathcal{A}_X}((\mathcal{A}^{loc}_X/\mathcal{A}_X)[-1], \mathcal{A}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} a_X^{-1}\, \mathrm{R}\Gamma(X, \mathcal{M})).$$

By adjunction, we have the morphism of functors $a_X^{-1} \circ \mathrm{R}\, a_{X*} \to \mathrm{id}$. It follows that we have a morphism $a_X^{-1}\, \mathrm{R}\Gamma(X, \mathcal{M}) \to \mathcal{M}$. Tensoring by $\mathcal{A}_X \overset{\mathrm{L}}{\underset{A_X}{\otimes}} \cdot$ we get

$$\mathcal{A}_X \otimes_{A_X} a_X^{-1}\, \mathrm{R}\Gamma(X, \mathcal{M}) \to \mathcal{A}_X \otimes_{A_X} \mathcal{M}. \tag{2.4.5}$$

Moreover,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}_X}(\mathcal{A}_X \otimes_{A_X} \mathcal{M}, \mathcal{M}) &\simeq \mathrm{Hom}_{A_X}(\mathcal{M}, \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_X, \mathcal{M})) \\ &\simeq \mathrm{Hom}_{A_X}(\mathcal{M}, \mathcal{M}).\end{aligned}$$

Consequently the image of the identity gives a morphism $\mathcal{A}_X \otimes_{A_X} \mathcal{M} \to \mathcal{M}$. By composing with (2.4.5), one obtains a morphism

$$\mathcal{A}_X \otimes_{A_X} a_X^{-1}\, \mathrm{R}\Gamma(X, \mathcal{M}) \to \mathcal{M}.$$

Applying the functor $(\cdot)^{\mathrm{cc}}$ to the preceding morphism we obtain

$$(\mathcal{A}_X \otimes_{A_X} a_X^{-1}\, \mathrm{R}\Gamma(X, \mathcal{M}))^{\mathrm{cc}} \to \mathcal{M}^{\mathrm{cc}}.$$

Since $\mathcal{M}$ is cohomologically complete, $\mathcal{M}^{\mathrm{cc}} \simeq \mathcal{M}$. Thus

$$(\mathcal{A}_X \otimes_{A_X} a_X^{-1} \mathrm{R}\Gamma(X, \mathcal{M}))^{\mathrm{cc}} \to \mathcal{M}. \tag{2.4.6}$$

Applying $\mathrm{gr}_\hbar$ to the preceding formula, and using the well known equivalence

$$\mathsf{D}^+_{\mathrm{qcoh}}(\mathcal{O}_X) \underset{\mathcal{O}_X \otimes_{B_X} -}{\overset{\mathrm{R}\Gamma(X,\cdot)}{\rightleftarrows}} \mathsf{D}^+(B) \;,$$

we obtain the isomorphism

$$\mathcal{O}_X \otimes_{B_X} \mathrm{R}\Gamma(X, \mathrm{gr}_\hbar \mathcal{M}) \xrightarrow{\sim} \mathrm{gr}_\hbar \mathcal{M}.$$

Since $(\mathcal{A}_X \otimes_{A_X} a_X^{-1} \mathrm{R}\Gamma(X, \mathcal{M}))^{\mathrm{cc}}$ and $\mathcal{M}$ are cohomologically complete modules, it follows that (2.4.6) is an isomorphism.

Let $M \in \mathsf{D}^+_{\mathrm{cc}}(A)$. By definition,

$$\Phi \circ \Psi(M) = \mathrm{R}\Gamma(X, (\mathcal{A}_X \otimes_{A_X} a_X^{-1} M)^{\mathrm{cc}}).$$

and using Lemma 2.4.2 we get that

$$\Phi \circ \Psi(M) \simeq (\mathrm{R}\Gamma(X, \mathcal{A}_X \otimes_{A_X} M))^{\mathrm{cc}}.$$

We have a morphism

$$\mathrm{R}a_{X*}(\mathcal{A}_X) \otimes_A M \to \mathrm{R}\, a_{X*}(\mathcal{A}_X \otimes_{A_X} a_X^{-1} M).$$

Since X is affine we obtain $\mathrm{R}\, a_{X*}\mathcal{A}_X \simeq A$ thus

$$M \to \mathrm{R}\Gamma(X, \mathcal{A}_X \otimes_{A_X} M).$$

We have a map

$$M^{\mathrm{cc}} \to (\mathrm{R}\Gamma(X, \mathcal{A}_X \otimes_{A_X} M))^{\mathrm{cc}}. \tag{2.4.7}$$

Applying the functor $\mathrm{gr}_\hbar$, we obtain

$$\mathrm{gr}_\hbar M \to \mathrm{R}\Gamma(X, \mathcal{O}_X \otimes_{B_X} \mathrm{gr}_\hbar M). \tag{2.4.8}$$

Using Lemma 2.4.3, we deduce that the map (2.4.8) is an isomorphism. It follows by Corollary 2.2.26 that the morphism (2.4.7) is an isomorphism. This proves the announced equivalence. □

Chapter 3

The Lefschetz-Lunts formula for deformation quantization modules

3.1 Introduction

Inspired by the work of D. Shklyarov (see [Shk07a]), V. Lunts has established in [Lun11] a Lefschetz type formula which calculates the trace of a coherent kernel acting on the Hochschild homology of a projective variety (Theorem 3.4.7). This result has inspired several other works ([CT11, Pol11]). In [CT11], Cisinski and Tabuada recover the result of Lunts via the theory of non-commutative motives. In [Pol11], Polischuk proves similar formulas and applies them to matrix factorisation. The aim of this chapter, extracted from [Pet10], is to adapt Lunts formula to the case of deformation quantization modules (DQ-modules) of Kashiwara-Schapira on complex Poisson manifolds. For that purpose, we develop an abstract framework which allows one to obtain Lefschetz-Lunts type formulas in symmetric monoidal categories endowed with some additional data.

Our proof relies essentially on two facts. The first one is that the composition operation on the Hochschild homology is compatible in some sense with the symmetric monoidal structures of the categories involved. The second one is the functoriality of the Hochschild class with respect to composition of kernels. This suggest that the Lefeschtz-Lunts formula is a 2-categorical statement and that it might be possible to build a set-up, in the spirit of [CW10], which would encompass simultaneously these two aspects.

Let us compare briefly the different approaches and settings of [Lun11], [CT11] and [Pol11] to ours. As already mentioned, we are working in the framework of deformation quantization modules over complex manifolds.

The approach of Lunts is based on a certain list of properties of the Hochschild homology of algebraic varieties (see [Lun11, §3]). These properties mainly concern the behaviour of Hochschild homology with respect to the composition of kernels and its functoriality. A straightforward consequence of these properties is that the morphism $X \to \mathrm{pt}$ induces a map from the Hochschild homology of X to the ground field k. Such a map does not exist in the theory of DQ-modules. Thus, it is not possible to integrate a single class with values in Hochschild homology and one has to integrate a pair of classes. Then, it seems that the method of V. Lunts cannot be carried out in our context.

In [CT11], the authors showed that the results of V. Lunts for projective varieties can be derived from a very general statement for additive invariants of smooth and proper differential graded category in the sense of Kontsevich. However, it is not clear that this approach would work for DQ-modules even in the algebraic case. Indeed, the results used to relate non-commutative motives to more classical geometric objects rely on the existence of a compact generator for the derived category of quasi-coherent sheaves which is a classical generator of the derived category of coherent sheaves. To the best of our knowledge, there are no such results for DQ-modules. Similarly, the approach of [Pol11] does not seem to be applicable to DQ-modules.

The chapter is organised as follow. In the first part, we sketch a formal framework in which we can get a formula for the trace of a class acting on a certain homology, starting from a symmetric monoidal category endowed with some specific data. In the second part, we briefly review, following [KS12], some elements of the theory of DQ-modules. The last part is mainly devoted to the proof of the Lefschetz-Lunts theorems for DQ-modules. Then, we briefly explain how to recover some of Lunts's results.

3.2 A general framework for Lefschetz type theorems

3.2.1 A few facts about symmetric monoidal categories and traces

In this subsection, we recall a few classical facts concerning dual pairs and traces in symmetric monoidal categories. References for this subsection are [KS06, Chap.4], [LMSM86], [PS11].

Let $\mathscr{C}$ be a symmetric monoidal category with product $\otimes$, unit object $\mathbf{1}_{\mathscr{C}}$ and symmetry isomorphism σ. All along this chapter, we identify $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$.

Definition 3.2.1. We say that $X \in \mathrm{Ob}(\mathscr{C})$ is dualizable if there is $Y \in \mathrm{Ob}(\mathscr{C})$ and two morphisms, $\eta : \mathbf{1}_{\mathscr{C}} \to X \otimes Y$, $\varepsilon : Y \otimes X \to \mathbf{1}_{\mathscr{C}}$ called coevaluation and evaluation such that the condition (a) and (b) are satisfied:

(a) The composition $X \simeq \mathbf{1}_{\mathscr{C}} \otimes X \overset{\eta\otimes \mathrm{id}_X}{\to} X \otimes Y \otimes X \overset{\mathrm{id}_X \otimes \varepsilon}{\to} X \otimes \mathbf{1}_{\mathscr{C}} \simeq X$ is the identity of X.

(b) The composition $Y \simeq Y \otimes \mathbf{1}_{\mathscr{C}} \overset{\mathrm{id}_Y \otimes \eta}{\to} Y \otimes X \otimes Y \overset{\varepsilon \otimes \mathrm{id}_Y}{\to} \mathbf{1}_{\mathscr{C}} \otimes Y \simeq Y$ is the identity of Y.

We call Y a dual of X and say that (X, Y) is a dual pair.

We shall prove that some diagrams commute. For that purpose recall the useful lemma below communicated to us by Masaki Kashiwara.

Lemma 3.2.2. *Let $\mathscr{C}$ be a monoidal category with unit. Let (X, Y) be a dual pair with coevaluation and evaluation morphisms*

$$\mathbf{1}_{\mathscr{C}} \overset{\eta}{\to} X \otimes Y,\ Y \otimes X \overset{\varepsilon}{\to} \mathbf{1}_{\mathscr{C}}.$$

Let $f : \mathbf{1}_{\mathscr{C}} \to X \otimes Y$ be a morphism such that $(\mathrm{id}_X \otimes \varepsilon) \circ (f \otimes \mathrm{id}_X) = \mathrm{id}_X$. Then $f = \eta$.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathbf{1}_{\mathscr{C}} \arrow[r, "\eta"] \arrow[d, "f"'] & X \otimes Y \arrow[d, "f\otimes \mathrm{id}_X \otimes \mathrm{id}_Y"] & \\
X \otimes Y \arrow[r, "\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \eta"'] & X \otimes Y \otimes X \otimes Y \arrow[dr, "\mathrm{id}_X \otimes \varepsilon \otimes \mathrm{id}_Y"] & \\
& & X \otimes Y.
\end{tikzcd}$$

By the hypothesis, $(\mathrm{id}_X \otimes \varepsilon \otimes \mathrm{id}_Y) \circ (f \otimes \mathrm{id}_X \otimes \mathrm{id}_Y) = \mathrm{id}_X \otimes \mathrm{id}_Y$ and $(\mathrm{id}_X \otimes \varepsilon \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \eta) = (\mathrm{id}_X \otimes \mathrm{id}_Y)$. Therefore, $\eta = f$. □

The next proposition is well known. But, we do know not the original reference. A proof can be found in [KS06, Chap.4].

Proposition 3.2.3. *If (X, Y) is a dual pair, then for every Z, $W \in \mathrm{Ob}(\mathscr{C})$, there are natural isomorphisms*

$$\Phi : \mathrm{Hom}_{\mathscr{C}}(Z, W \otimes Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(Z \otimes X, W),$$
$$\Psi : \mathrm{Hom}_{\mathscr{C}}(Y \otimes Z, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(Z, X \otimes W)$$

where for $f \in \mathrm{Hom}_{\mathscr{C}}(Z, W \otimes Y)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(Y \otimes Z, W)$,

$$\Phi(f) = (\mathrm{id}_W \otimes \varepsilon) \circ (f \otimes \mathrm{id}_X),$$
$$\Psi(g) = (\mathrm{id}_X \otimes g) \circ (\eta \otimes \mathrm{id}_Z).$$

Remark 3.2.4. It follows that Y is a representative of the functor $Z \mapsto \mathrm{Hom}_{\mathscr{C}}(Z \otimes X, \mathbf{1}_{\mathscr{C}})$ as well as a representative of the functor $W \mapsto \mathrm{Hom}_{\mathscr{C}}(\mathbf{1}_{\mathscr{C}}, X \otimes W)$. Therefore, the dual of a dualizable object is unique up to a unique isomorphism.

Definition 3.2.5. For a dualizable object X, the trace of $f : X \to X$ denoted $\mathrm{Tr}(f)$ is the composition

$$\mathbf{1}_{\mathscr{C}} \to X \otimes Y \xrightarrow{f \otimes \mathrm{id}} X \otimes Y \xrightarrow{\sigma} Y \otimes X \xrightarrow{\varepsilon} \mathbf{1}_{\mathscr{C}}.$$

Then, $\mathrm{Tr}(f) \in \mathrm{Hom}_{\mathscr{C}}(\mathbf{1}_{\mathscr{C}}, \mathbf{1}_{\mathscr{C}})$.

Remark 3.2.6. The trace could also by defined as the following composition

$$\mathbf{1}_{\mathscr{C}} \to X \otimes Y \xrightarrow{\sigma} Y \otimes X \xrightarrow{\mathrm{id} \otimes f} Y \otimes X \xrightarrow{\varepsilon} \mathbf{1}_{\mathscr{C}}.$$

These two definitions of the trace coincide because $(\mathrm{id} \otimes f)\sigma = \sigma(f \otimes \mathrm{id})$ since σ is a natural transformation.

Recall the following fact.

Lemma 3.2.7. *With the notation of Definition 3.2.5, the trace is independent of the choice of a dual for X.*

Proof. Let Y and Y' two duals of X with evaluations ε, ε' and coevalution η and η'. By definition of a representative of the functor $Z \mapsto \mathrm{Hom}_{\mathscr{C}}(Z \otimes X, \mathbf{1}_{\mathscr{C}})$ there exist a unique isomorphism $\theta : Y \to Y'$ such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(Z, Y') \arrow[r, "\Phi'"] & \mathrm{Hom}_{\mathscr{C}}(Z \otimes X, \mathbf{1}_{\mathscr{C}}) \\
\mathrm{Hom}_{\mathscr{C}}(Z, Y) \arrow[u, "\theta\circ"] \arrow[ur, "\Phi"']
\end{tikzcd}.$$

commutes. For $Z = Y$, the diagram, applied to id_Y, implies $\varepsilon = \varepsilon' \circ (\theta \otimes \mathrm{id}_X)$. Using Lemma 3.2.2, we get that $\eta = (\mathrm{id}_X \otimes \theta^{-1}) \circ \eta'$. It follows that the diagram

$$\begin{tikzcd}
& X \otimes Y \arrow[r, "f\otimes\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes\theta"] & X \otimes Y \arrow[r, "\sigma"] \arrow[dd, "\mathrm{id}\otimes\theta"] & Y \otimes X \arrow[dr, "\varepsilon"] \arrow[dd, "\theta\otimes\mathrm{id}"] & \\
\mathbf{1}_{\mathscr{C}} \arrow[ur, "\eta"] \arrow[dr, "\eta'"'] & & & & \mathbf{1}_{\mathscr{C}} \\
& X \otimes Y' \arrow[r, "f\otimes\mathrm{id}"'] & X \otimes Y' \arrow[r, "\sigma"'] & Y' \otimes X \arrow[ur, "\varepsilon'"'] &
\end{tikzcd}$$

commutes which proves the claim. ∎

Example 3.2.8. (see [LMSM86, Chap.3]) Let k be a Noetherian commutative ring of finite weak global dimension. Let $\mathsf{D}^b(k)$ be the bounded derived category of the category of k-modules. It is a symmetric monoidal category for $\overset{\mathrm{L}}{\underset{k}{\otimes}}$. We denote by τ the natural commutativity isomorphism of $\mathsf{D}(k)$ and by $\mathsf{D}^b_f(k)$, the full subcategory of $\mathsf{D}^b(k)$ whose objects are the complexes with finite type cohomology. If $M \in \mathrm{Ob}(\mathsf{D}^b_f(k))$, its dual is given by $\mathrm{RHom}_k(M,k)$. The evaluation and the coevaluation are given by

$$\mathrm{ev} : \mathrm{RHom}_k(M,k) \overset{\mathrm{L}}{\underset{k}{\otimes}} M \to k$$
$$\mathrm{coev} : k \to \mathrm{RHom}_k(M,M) \xleftarrow{\sim} M \overset{\mathrm{L}}{\underset{k}{\otimes}} \mathrm{RHom}_k(M,k).$$

If we further assume that k is an integral domain, then k can be embedded into its field of fraction $\mathrm{F}(k)$. If f is an endomorphism of M then the trace of f

$$k \overset{\mathrm{coev}}{\to} M \otimes \mathrm{RHom}_k(M,k) \overset{f\otimes\mathrm{id}}{\to} M \otimes \mathrm{RHom}_k(M,k) \overset{\tau}{\to} \mathrm{RHom}_k(M,k) \otimes M \overset{\mathrm{ev}}{\to} k$$

coincides with $\sum_i (-1)^i \operatorname{Tr}(\mathrm{H}^i(\mathrm{id}_{\mathrm{F}(k)} \otimes f))$. If $f = \mathrm{id}_M$, one sets

$$\chi(M) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathrm{F}(k)}(\mathrm{H}^i(M)).$$

3.2.2 The framework

In this section, we define a general framework for Lefschetz-Lunts type theorems. Let $\mathscr{C}$ be a symmetric monoidal category with product $\otimes$, unit object $\mathbf{1}_{\mathscr{C}}$ and symmetry isomorphism σ. Let k be a Noetherian commutative ring with finite cohomological dimension.

Assume we are given:

(a) a monoidal functor $(\cdot)^a : \mathscr{C} \to \mathscr{C}$ such that $(\cdot)^a \circ (\cdot)^a = \mathrm{id}_{\mathscr{C}}$ and $\mathbf{1}^a_{\mathscr{C}} \simeq \mathbf{1}_{\mathscr{C}}$

(b) a symmetric monoidal functor $(L, \mathfrak{K}) : \mathscr{C} \to \mathsf{D}^b(k)$ where $\mathfrak{K}$ is the isomorphism of bifunctor from $L(\cdot) \overset{\mathrm{L}}{\otimes} L(\cdot)$ to $L(\cdot \otimes \cdot)$. That is $L(X) \overset{\mathrm{L}}{\otimes} L(Y) \overset{\mathfrak{K}}{\simeq} L(X \otimes Y)$ naturally in X and Y and $L(\mathbf{1}_{\mathscr{C}}) \simeq k$,

(c) for $X_i \in \mathrm{Ob}(\mathscr{C})$ $(i = 1, 2, 3)$, a morphism

$$\underset{2}{\cup} : L(X_1 \otimes X_2^a) \overset{\mathrm{L}}{\otimes} L(X_2 \otimes X_3^a) \to L(X_1 \otimes X_3^a),$$

(d) for every $X \in \mathrm{Ob}(\mathscr{C})$, a morphism

$$L_{\Delta_X} : k \to L(X \otimes X^a),$$

these data verifying the following properties:

(P1) for $X_1,\ X_3 \in \mathrm{Ob}(\mathscr{C})$, the diagram

$$\begin{tikzcd}
L(X_1 \otimes \mathbf{1}^a_{\mathscr{C}}) \overset{\mathrm{L}}{\otimes} L(\mathbf{1}_{\mathscr{C}} \otimes X_3) \arrow[r, "\underset{\mathbf{1}_{\mathscr{C}}}{\cup}"] \arrow[d, "\wr"] & L(X_1 \otimes X_3) \arrow[d, "\mathrm{id}"] \\
L(X_1) \overset{\mathrm{L}}{\otimes} L(X_3) \arrow[r, "\mathfrak{K}"] & L(X_1 \otimes X_3)
\end{tikzcd}$$

commutes,

(P2) for $X_1,\ X_2,\ X_3,\ X_4 \in \mathrm{Ob}(\mathscr{C})$, the diagram

$$\begin{tikzcd} L(X_1 \otimes X_2^a) \overset{\mathrm{L}}{\otimes} L(X_2 \otimes X_3^a) \overset{\mathrm{L}}{\otimes} L(X_3 \otimes X_4^a) \arrow[r, "\underset{2}{\cup} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \underset{3}{\cup}"'] & L(X_1 \otimes X_3^a) \overset{\mathrm{L}}{\otimes} L(X_3 \otimes X_4) \arrow[d, "\underset{3}{\cup}"] \\ L(X_1 \otimes X_2^a) \overset{\mathrm{L}}{\otimes} L(X_2 \otimes X_4^a) \arrow[r, "\underset{2}{\cup}"] & L(X_1 \otimes X_4^a) \end{tikzcd}$$

commutes,

(P3) the diagram

$$\begin{tikzcd} k \arrow[r, "L_{\Delta_X}"] \arrow[dr, "L_{\Delta_{X^a}}"'] & L(X \otimes X^a) \arrow[d, "L(\sigma)"] \\ & L(X^a \otimes X) \end{tikzcd}$$

commutes,

(P4) the composition

$$L(X) \overset{L_{\Delta_X} \otimes \mathrm{id}_{L(X)}}{\longrightarrow} L(X \otimes X^a) \overset{\mathrm{L}}{\otimes} L(X) \overset{\underset{X}{\cup}}{\to} L(X)$$

is the identity of $L(X)$ and the composition

$$L(X^a) \overset{\mathrm{id}_{L(X^a)} \otimes L_{\Delta_X}}{\longrightarrow} L(X^a) \overset{\mathrm{L}}{\otimes} L(X \otimes X^a) \overset{\underset{X^a}{\cup}}{\to} L(X^a)$$

is the identity of $L(X^a)$,

(P5) the diagram

$$\begin{tikzcd} L(X \otimes X^a) \overset{\mathrm{L}}{\otimes} L(X^a \otimes X) \arrow[r, "\underset{X^a \otimes X}{\cup}"] & k \\ L(X^a) \overset{\mathrm{L}}{\otimes} L(X). \arrow[u, "L_{\Delta_X} \otimes \mathfrak{K}"] \arrow[ur, "\underset{X}{\cup}"'] & \end{tikzcd}$$

commutes,

(P6) for X_1 and X_2 belonging to $\mathrm{Ob}(\mathscr{C})$, the diagram

$$\begin{tikzcd} L((X_1 \otimes X_2)^a) \overset{\mathrm{L}}{\otimes} L((X_1 \otimes X_2)) \arrow[r, "\underset{X_1 \otimes X_2}{\cup}"] & k \\ L((X_2 \otimes X_1)^a) \overset{\mathrm{L}}{\otimes} L(X_2 \otimes X_1) \arrow[u, "L(\sigma) \otimes L(\sigma)"] \arrow[ur, "\underset{X_2 \otimes X_1}{\cup}"'] & \end{tikzcd}$$

commutes.

Lemma 3.2.9. *The object $L(X^a)$ is a dual of $L(X)$ with coevalution $\eta := \mathfrak{K}^{-1} \circ L_{\Delta_X}$ and evaluation $\varepsilon := \underset{X}{\cup} : L(X^a) \overset{\mathrm{L}}{\otimes} L(X) \to k$.*

Proof. Consider the diagram

$$\begin{tikzcd} L(X) \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, equal] & L(X)\overset{\mathrm{L}}{\otimes}L(X^a)\overset{\mathrm{L}}{\otimes}L(X) \arrow[r, "\mathrm{id}\otimes\varepsilon"] \arrow[d, "\mathfrak{K}\otimes\mathrm{id}"] & L(X) \arrow[d, equal] \\ L(X) \arrow[r, "L_{\Delta_X}\otimes\mathrm{id}"'] & L(X\otimes X^a)\overset{\mathrm{L}}{\otimes}L(X) \arrow[r, "\underset{X^a}{\cup}"'] & L(X) \end{tikzcd}$$

and the diagram

$$\begin{tikzcd} L(X^a) \arrow[r, "\mathrm{id}\otimes\eta"] \arrow[d, equal] & L(X^a)\overset{\mathrm{L}}{\otimes}L(X)\overset{\mathrm{L}}{\otimes}L(X^a) \arrow[r, "\varepsilon\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathfrak{K}"] & L(X^a) \arrow[d, equal] \\ L(X^a) \arrow[r, "\mathrm{id}\otimes L_{\Delta_{X^a}}"'] & L(X^a)\overset{\mathrm{L}}{\otimes}L(X\otimes X^a) \arrow[r, "\underset{X}{\cup}"'] & L(X^a). \end{tikzcd}$$

These diagrams are made of two squares. The left squares commute by definition of η. The squares on the right commute because of the Property (P2). It follows that the two diagrams commute. Property (P4) implies that the bottom line of each diagram is equal to the identity. This proves the proposition. □

The preceding lemma shows that $L(X)$ is a dualizable object of $\mathsf{D}^b(k)$. We set $L(X)^\star = \mathrm{RHom}_k(L(X), k)$. By Remark 3.2.4, we have $L(X)^\star \simeq L(X^a)$.

Let $\lambda : k \to L(X\otimes X^a)$ be a morphism of $\mathsf{D}^b(k)$. It defines a morphism

$$\Phi_\lambda : L(X) \xrightarrow{\lambda\overset{\mathrm{L}}{\otimes}\mathrm{id}} L(X\otimes X^a)\overset{\mathrm{L}}{\otimes}L(X) \xrightarrow{\underset{X}{\cup}} L(X). \tag{3.2.1}$$

Consider the diagram

$$\begin{tikzcd}[column sep=small] & L(X)\overset{\mathrm{L}}{\otimes}L(X)^\star \arrow[r, "\Phi_\lambda\otimes\mathrm{id}"] & L(X)\overset{\mathrm{L}}{\otimes}L(X)^\star \arrow[r, "\tau"] & L(X)^\star\overset{\mathrm{L}}{\otimes}L(X) \arrow[dr, "\mathrm{ev}"] & \\ k \arrow[ur, "\mathrm{coev}"] \arrow[dr, "\eta"'] & & & & k \\ & L(X)\overset{\mathrm{L}}{\otimes}L(X^a) \arrow[r, "\Phi_\lambda\otimes\mathrm{id}"] & L(X)\overset{\mathrm{L}}{\otimes}L(X^a) \arrow[r, "\tau"] & L(X^a)\overset{\mathrm{L}}{\otimes}L(X) \arrow[ur, "\varepsilon"'] & \end{tikzcd} \tag{3.2.2}$$

Lemma 3.2.10. *The diagram (3.2.2) commutes.*

Proof. By Lemma 3.2.9, $L(X^a)$ is a dual of $L(X)$ with evaluation morphism ε and coevaluation morphism η. It follows from Lemma 3.2.7 that the diagram (3.2.2) commutes. □

We identify λ and the image of 1_k by λ and similarly for L_{Δ_X}. From now on, we write indifferently $\cup$ as a morphism or as an operation, as for example in Theorem 3.2.11.

Theorem 3.2.11. *Assuming properties (P1) to (P5), we have the formula*

$$\mathrm{Tr}(\Phi_\lambda) = L_{\Delta_X} \underset{X^a\otimes X}{\cup} L(\sigma)\lambda.$$

If we further assume Property (P6) we have the formula

$$\mathrm{Tr}(\Phi_\lambda) = L_{\Delta_{X^a}} \underset{X \otimes X^a}{\cup} \lambda.$$

Proof. By definition of Φ_λ, the diagram

$$\begin{tikzcd}
 & L(X) \overset{\mathrm{L}}{\otimes} L(X^a) \arrow[r, "\Phi_\lambda \otimes \mathrm{id}"] & L(X) \overset{\mathrm{L}}{\otimes} L(X^a) \arrow[r] & L(X^a) \overset{\mathrm{L}}{\otimes} L(X) \arrow[dr, "\varepsilon"] & \\
k \arrow[ur, "\eta"] \arrow[dr, "\lambda \otimes \eta"'] & & & & k \\
 & L(X \otimes X^a) \overset{\mathrm{L}}{\otimes} L(X) \overset{\mathrm{L}}{\otimes} L(X^a) \arrow[r, "\underset{X}{\cup} \otimes \mathrm{id}"] & L(X) \overset{\mathrm{L}}{\otimes} L(X^a) \arrow[r] & L(X^a) \overset{\mathrm{L}}{\otimes} L(X) \arrow[ur, "\varepsilon"'] &
\end{tikzcd} \qquad (3.2.3)$$

commutes.

Thus, computing the trace of Φ_λ is equivalent to compute the lower part of diagram (3.2.3).

We denote by ζ the map

$$\zeta : L(X^a \otimes X) \simeq k \overset{\mathrm{L}}{\otimes} L(X^a \otimes X) \overset{L_{\Delta_X} \otimes \mathrm{id}}{\to} L(X \otimes X^a) \overset{\mathrm{L}}{\otimes} L(X^a \otimes X) \overset{\underset{X^a \otimes X}{\cup}}{\to} k.$$

Consider the diagram

$$\begin{tikzcd}
 & k \arrow[dl, "\lambda \otimes \eta"'] \arrow[dr, "\lambda \otimes L_{\Delta_X}"] \arrow[d, phantom, "1"] & \\
L(X \otimes X^a) \overset{\mathrm{L}}{\otimes} L(X) \overset{\mathrm{L}}{\otimes} L(X^a) \arrow[rr, "\mathrm{id} \otimes \mathfrak{K}"] \arrow[d, "\underset{X}{\cup} \overset{\mathrm{L}}{\otimes} \mathrm{id}"'] & {} \arrow[d, phantom, "2"] & L(X \otimes X^a) \overset{\mathrm{L}}{\otimes} L(X \otimes X^a) \arrow[d, "\underset{X}{\cup}"] \\
L(X) \overset{\mathrm{L}}{\otimes} L(X^a) \arrow[rr, "\mathfrak{K}"] \arrow[d] & {} \arrow[d, phantom, "3"] & L(X \otimes X^a) \arrow[d, "L(\sigma)"] \\
L(X^a) \overset{\mathrm{L}}{\otimes} L(X) \arrow[rr, "\mathfrak{K}"] \arrow[dr, "\underset{X}{\cup}"'] & {} \arrow[d, phantom, "4"] & L(X^a \otimes X) \arrow[dl, "\zeta"] \\
 & k &
\end{tikzcd} \qquad (3.2.4)$$

This diagram is made of four sub-diagrams numbered from 1 to 4.

1. The sub-diagram 1 commutes by definition of η,
2. notice that $\mathfrak{K} = \underset{\mathbf{1}_{\mathscr{C}}}{\cup}$ by the Property (P1). Then the sub-diagram 2 commutes by the Property (P2),
3. the sub-diagram 3 commutes because L is a symmetric monoidal functor,
4. the sub-diagram 4 is the diagram of Property (P5).

Applying Property (P4), we find that the right side of the diagram (3.2.4) is equal to $L_{\Delta_X} \underset{X^a \otimes X}{\cup} L(\sigma)\lambda$.

By the Property (P6), $L_{\Delta_X} \underset{X^a \otimes X}{\cup} L(\sigma)\lambda = L(\sigma) L_{\Delta_X} \underset{X \otimes X^a}{\cup} \lambda$ and by the Property (P3), $L(\sigma) L_{\Delta_X} = L_{\Delta_{X^a}}$, the result follows. $\square$

3.3 A short review on DQ-modules

Deformation quantization modules have been introduced in [Kon01] and systematically studied in [KS12]. We shall first recall here the main features of this theory, following the notation of loc. cit.

In all this chapter, a manifold means a complex analytic manifold. We denote by $\mathbb{C}^\hbar$ the ring $\mathbb{C}[[\hbar]]$. A Deformation Quantization algebroid stack (DQ-algebroid for short) on a complex manifold X with structure sheaf $\mathcal{O}_X$, is a stack of $\mathbb{C}^\hbar$-algebras locally isomorphic to a star algebra $(\mathcal{O}_X[[\hbar]], \star)$. If $\mathcal{A}_X$ is a DQ-algebroid on a manifold X then the opposite DQ-algebroid $\mathcal{A}_X^{\mathrm{op}}$ is denoted by $\mathcal{A}_{X^a}$. The diagonal embedding is denoted by $\delta_X : X \to X \times X$.

If X and Y are two manifolds endowed with DQ-algebroids $\mathcal{A}_X$ and $\mathcal{A}_Y$, then $X \times Y$ is canonically endowed with the DQ-algebroid $\mathcal{A}_{X\times Y} := \mathcal{A}_X \boxtimes \mathcal{A}_Y$ (see [KS12, §2.3]). Following [KS12, §2.3], we denote by $\cdot \boxtimes \cdot$ is the exterior product and by $\cdot \underline{\boxtimes} \cdot$ the bifunctor $\mathcal{A}_{X\times Y} \underset{\mathcal{A}_X \boxtimes \mathcal{A}_Y}{\otimes} (\cdot \boxtimes \cdot)$:

$$\cdot \underline{\boxtimes} \cdot : \mathrm{Mod}(\mathcal{A}_X) \times \mathrm{Mod}(\mathcal{A}_Y) \to \mathrm{Mod}(\mathcal{A}_{X\times Y}).$$

We write $\cdot \overset{\mathrm{L}}{\underline{\boxtimes}} \cdot$ for the corresponding derived bifunctor.

We write $\mathcal{C}_X$ for the $\mathcal{A}_{X\times X^a}$-module $\delta_{X*}\mathcal{A}_X$ and $\omega_X \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_{X\times X^a})$ for the dualizing complex of DQ-modules. We denote by $\mathbb{D}'_{\mathcal{A}_X}$ the duality functor of $\mathcal{A}_X$-modules:

$$\mathbb{D}'_{\mathcal{A}_X}(\cdot) := \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_X}(\cdot, \mathcal{A}_X).$$

Consider complex manifolds X_i endowed with DQ-algebroids $\mathcal{A}_{X_i}$ $(i = 1, 2, \dots)$.

Notation 3.3.1. (i) Consider a product of manifolds $X_1 \times X_2 \times X_3$, we write it X_{123}. We denote by p_i the i-th projection and by p_{ij} the (i, j)-th projection (*e.g.*, p_{13} is the projection from $X_1 \times X_1^a \times X_2$ to $X_1 \times X_2$). We use similar notation for a product of four manifolds.

(ii) We write $\mathcal{A}_i$ and $\mathcal{A}_{ij^a}$ instead of $\mathcal{A}_{X_i}$ and $\mathcal{A}_{X_i \times X_j^a}$ and similarly with other products. We use the same notations for $\mathcal{C}_{X_i}$.

(iii) When there is no risk of confusion, we do note write the symbols p_i^{-1} and similarly with i replaced with ij, etc.

(iv) If $\mathcal{K}_1$ is an object of $\mathsf{D}^b(\mathbb{C}^\hbar_{12})$ and $\mathcal{K}_2$ is an object of $\mathsf{D}^b(\mathbb{C}^\hbar_{23})$, we write $\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2$ for $\mathrm{R}\, p_{13!}(p_{12}^{-1}\mathcal{K}_1 \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar_{123}}{\otimes}} p_{23}^{-1}\mathcal{K}_2)$.

(v) We write $\overset{\mathrm{L}}{\otimes}$ for the tensor product over $\mathbb{C}^\hbar$.

3.3.1 Hochschild homology

Let X be a complex manifold endowed with a DQ-algebroid $\mathcal{A}_X$. Recall that its Hochschild homology is defined by

$$\mathcal{HH}(\mathcal{A}_X) := \delta_X^{-1}(\mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X\times X^a}}{\otimes}} \mathcal{C}_X) \in \mathsf{D}^b(\mathbb{C}^\hbar_X).$$

We denote by $\mathbb{HH}(\mathcal{A}_X)$ the object $\mathrm{R}\Gamma(X, \mathcal{HH}(\mathcal{A}_X))$ of the category $\mathsf{D}^b(\mathbb{C}^\hbar)$ and by $\mathrm{HH}_0(\mathcal{A}_X)$ the $\mathbb{C}^\hbar$-module $\mathrm{H}^0(\mathbb{HH}(\mathcal{A}_X))$. We also set the notation, for a closed subset Λ of X, $\mathcal{HH}_\Lambda(\mathcal{A}_X) := \Gamma_\Lambda \mathcal{HH}(\mathcal{A}_X)$ and $\mathrm{HH}_{0,\Lambda}(\mathcal{A}_X) = \mathrm{H}^0(\mathrm{R}\Gamma_\Lambda(X; \mathcal{HH}(\mathcal{A}_X)))$.

Proposition 3.3.2. *There is a natural isomorphism*

$$\mathcal{HH}(\mathcal{A}_X) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{A}_{X\times X^a}}(\omega_X^{-1}, \mathcal{C}_X). \tag{3.3.1}$$

Proof. See [KS12, §4.1, p.103]. □

Remark 3.3.3. There is also a natural isomorphism

$$\mathcal{HH}(\mathcal{A}_X) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{A}_{X\times X^a}}(\mathcal{C}_X, \omega_X).$$

It can be obtain from the isomorphism (3.3.1) by adjunction.

Proposition 3.3.4 (Künneth isomorphism)**.** *Let X_i $(i = 1, 2)$ be complexe manifolds endowed with DQ-algebroids $\mathcal{A}_i$.*

(i) There is a natural morphism

$$\mathrm{R}\mathcal{H}om_{\mathcal{A}_{11^a}}(\omega_1^{-1}, \mathcal{C}_1) \overset{\mathrm{L}}{\boxtimes} \mathrm{R}\mathcal{H}om_{\mathcal{A}_{22^a}}(\omega_2^{-1}, \mathcal{C}_2) \to \mathrm{R}\mathcal{H}om_{\mathcal{A}_{121^a2^a}}(\omega_{12}^{-1}, \mathcal{C}_{12}). \tag{3.3.2}$$

(ii) If X_1 or X_2 is compact, this morphism induces a natural isomorphism

$$\mathfrak{K} : \mathbb{HH}(\mathcal{A}_1) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_2) \xrightarrow{\sim} \mathbb{HH}(\mathcal{A}_{12}). \tag{3.3.3}$$

Proof. (i) is clear.

(ii) By [KS12, Proposition 1.5.10] and [KS12, Proposition 1.5.12], the modules $\mathbb{HH}(\mathcal{A}_i)$ for $(i = 1, 2)$ and $\mathbb{HH}(\mathcal{A}_{12})$ are cohomologically complete. If X_1 is compact, then the $\mathbb{C}^\hbar$-module $\mathbb{HH}(\mathcal{A}_1)$ belongs to $\mathsf{D}^b_f(\mathbb{C}^\hbar)$. Thus, the $\mathbb{C}^\hbar$-module $\mathbb{HH}(\mathcal{A}_1) \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} \mathbb{HH}(\mathcal{A}_2)$ is still a cohomologically complete module (see [KS12, Proposition 1.6.5]).

Applying the functor $\mathrm{gr}_\hbar$ to the morphism (3.3.3), we obtain the usual Künneth isomorphism for Hochschild homology of complex manifolds. Since $\mathrm{gr}_\hbar$ is a conservative functor on the category of cohomologically complete modules, the morphism (3.3.3) is an isomorphism. □

3.3.2 Composition of Hochschild homology

Let Λ_{ij} $(i = 1, 2, j = i + 1)$ be a closed subset of X_{ij} and consider the hypothesis

$$p_{13} \text{ is proper on } \Lambda_{12} \times_{X_2} \Lambda_{23}. \tag{3.3.4}$$

We also set $\Lambda_{12} \circ \Lambda_{23} = p_{13}(p_{12}^{-1}\Lambda_{12} \cap p_{23}^{-1}\Lambda_{23})$.

Recall Proposition 4.2.1 of [KS12].

Proposition 3.3.5. *Let Λ_{ij} $(i = 1, 2\ j = i + 1)$ satisfying (3.3.4). There is a morphism*

$$\mathcal{HH}(\mathcal{A}_{12^a}) \underset{2}{\circ} \mathcal{HH}(\mathcal{A}_{23^a}) \to \mathcal{HH}(\mathcal{A}_{13^a}). \tag{3.3.5}$$

which induces a composition morphism for global sections

$$\underset{2}{\cup} : \mathbb{HH}_{\Lambda_{12}}(\mathcal{A}_{12^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}_{\Lambda_{23}}(\mathcal{A}_{23^a}) \to \mathbb{HH}_{\Lambda_{12}\circ\Lambda_{23}}(\mathcal{A}_{13^a}). \tag{3.3.6}$$

Corollary 3.3.6. *The morphism (3.3.5) induces a morphism*

$$\underset{\mathrm{pt}}{\cup} : \mathcal{HH}(\mathcal{A}_1) \overset{\mathrm{L}}{\boxtimes} \mathcal{HH}(\mathcal{A}_2) \to \mathcal{HH}(\mathcal{A}_{12}) \tag{3.3.7}$$

which coincides with the morphism (3.3.2).

Proof. The result follows directly from the construction of morphism (3.3.5). We refer the reader to [KS12, §4.2] for the construction. □

We will state a result concerning the associativity of the composition of Hochschild homology. It is possible to compose kernels in the framework of DQ-modules. Here, we identify $X_1 \times X_2 \times X_{3^a}$ with the diagonal subset of $X_1 \times X_{2^a} \times X_2 \times X_{3^a}$.

The following definition is Defininition 3.1.2 and Definition 3.1.3 of [KS12].

Definition 3.3.7. Let $\mathcal{K}_i \in \mathsf{D}^b(\mathcal{A}_{ij^a})$ $(i = 1, 2,\ j = i + 1)$. One sets

$$\begin{aligned}
\mathcal{K}_1 \underset{\mathcal{A}_2}{\overset{\mathrm{L}}{\otimes}} \mathcal{K}_2 &= (\mathcal{K}_1 \overset{\mathrm{L}}{\boxtimes} \mathcal{K}_2) \underset{\mathcal{A}_{22^a}}{\overset{\mathrm{L}}{\otimes}} \mathcal{C}_{X_2} \\
&= p_{12}^{-1}\mathcal{K}_1 \underset{p_{12}^{-1}\mathcal{A}_{1^a2}}{\overset{\mathrm{L}}{\otimes}} \mathcal{A}_{123} \underset{p_{23^a}^{-1}\mathcal{A}_{23^a}}{\overset{\mathrm{L}}{\otimes}} p_{23}^{-1}\mathcal{K}_2, \\
\mathcal{K}_1 \underset{X_2}{\circ} \mathcal{K}_2 &= \mathrm{R}p_{14!}((\mathcal{K}_1 \overset{\mathrm{L}}{\boxtimes} \mathcal{K}_2) \underset{\mathcal{A}_{22^a}}{\overset{\mathrm{L}}{\otimes}} \mathcal{C}_{X_2}), \\
\mathcal{K}_1 \underset{X_2}{*} \mathcal{K}_2 &= \mathrm{R}p_{14*}((\mathcal{K}_1 \overset{\mathrm{L}}{\boxtimes} \mathcal{K}_2) \underset{\mathcal{A}_{22^a}}{\overset{\mathrm{L}}{\otimes}} \mathcal{C}_{X_2}).
\end{aligned}$$

It should be noticed that $\underline{\overset{\mathrm{L}}{\otimes}}$, $\circ$ and $*$ are not associative in general.

Remark 3.3.8. There is a morphism $\mathcal{K}_1 \underset{\mathcal{A}_2}{\overset{\mathrm{L}}{\otimes}} \mathcal{K}_2 \to \mathcal{K}_1 \underset{\mathcal{A}_2}{\underline{\overset{\mathrm{L}}{\otimes}}} \mathcal{K}_2$ which is an isomorphism if $X_1 = \mathrm{pt}$ or $X_3 = \mathrm{pt}$.

The following proposition, which corresponds to [KS12, Proposition 3.2.4], states a result concerning the associativity of the composition of kernels in the category of DQ-modules and will be useful for the sketch of proof of Proposition 3.3.10.

Proposition 3.3.9. *Let* $\mathcal{K}_i \in \mathsf{D}^b_{coh}(\mathcal{A}_{i(i+1)^a})$ $(i = 1, 2, 3)$ *and let* $\mathcal{L} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_4)$. *Set* $\Lambda_i = \mathrm{Supp}(\mathcal{K}_i)$ *and assume that* $\Lambda_i \times_{X_{i+1}} \Lambda_{i+1}$ *is proper over* $X_i \times X_{i+2}$ *(i=1, 2).*

(i) There is a canonical isomorphism $(\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2) \overset{\mathrm{L}}{\boxtimes} \mathcal{L} \xrightarrow{\sim} \mathcal{K}_1 \underset{2}{\circ} (\mathcal{K}_2 \overset{\mathrm{L}}{\boxtimes} \mathcal{L})$.

(ii) There are canonical isomorphisms

$$(\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2) \underset{3}{\circ} \mathcal{K}_3 \xleftarrow{\sim} (\mathcal{K}_1 \overset{\mathrm{L}}{\boxtimes} \mathcal{K}_2 \overset{\mathrm{L}}{\boxtimes} \mathcal{K}_3) \underset{22^a33^a}{\circ} (\mathcal{C}_2 \overset{\mathrm{L}}{\boxtimes} \mathcal{C}_3) \xrightarrow{\sim} \mathcal{K}_1 \underset{2}{\circ} (\mathcal{K}_2 \underset{3}{\circ} \mathcal{K}_3).$$

The next proposition is the translation of Property (P2) in the framework of DQ-modules.

Proposition 3.3.10. *(i) Assume that X_i is compact for $i = 2,\ 3$. The following diagram is commutative*

$$\begin{tikzcd}
\mathcal{HH}(\mathcal{A}_{12^a}) \underset{2}{\circ} \mathcal{HH}(\mathcal{A}_{23^a}) \underset{3}{\circ} \mathcal{HH}(\mathcal{A}_{34^a}) \arrow[r] \arrow[d] & \mathcal{HH}(\mathcal{A}_{12^a}) \underset{2}{\circ} \mathcal{HH}(\mathcal{A}_{24^a}) \arrow[d] \\
\mathcal{HH}(\mathcal{A}_{13^a}) \underset{3}{\circ} \mathcal{HH}(\mathcal{A}_{34^a}) \arrow[r] & \mathcal{HH}(\mathcal{A}_{14^a}).
\end{tikzcd}$$

(ii) Assume that X_i is compact for $i = 1,\ 2,\ 3,\ 4$. The preceding diagram induces a commutative diagram

$$\begin{tikzcd}
\mathbb{HH}(\mathcal{A}_{12^a}) \overset{L}{\otimes} \mathbb{HH}(\mathcal{A}_{23^a}) \overset{L}{\otimes} \mathbb{HH}(\mathcal{A}_{34^a}) \arrow[r] \arrow[d] & \mathbb{HH}(\mathcal{A}_{12^a}) \overset{L}{\otimes} \mathbb{HH}(\mathcal{A}_{24^a}) \arrow[d] \\
\mathbb{HH}(\mathcal{A}_{13^a}) \overset{L}{\otimes} \mathbb{HH}(\mathcal{A}_{34^a}) \arrow[r] & \mathbb{HH}(\mathcal{A}_{14^a}).
\end{tikzcd}$$

Sketch of Proof. (i) If $\mathcal{M} \in \mathsf{D}(\mathcal{A}_X)$ and $\mathcal{N} \in \mathsf{D}(\mathcal{A}_Y)$, we write $\mathcal{M}\mathcal{N}$ for $\mathcal{M}\overset{L}{\underline{\boxtimes}}\mathcal{N}$ and i^k for $\underbrace{X_i \times \ldots \times X_i}_{k\ times}$. For the legibility, we omit the upper script $(\cdot)^a$ when indicating the base of a composition.

Following the notation of [KS12, §4.2], we set $S_{ij} := \omega_i^{-1}\overset{L}{\underline{\boxtimes}}\mathcal{C}_{j^a} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{ii^aj^aj})$ and $K_{ij} = \mathcal{C}_i\overset{L}{\underline{\boxtimes}}\omega_{j^a} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{ii^aj^aj})$. It follows that

$$\mathcal{HH}(\mathcal{A}_{ij^a}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{A}_{ii^aj^aj}}(S_{ij}, K_{ij}).$$

We deduce from Proposition 3.3.9 (ii), the following diagram which commutes.

$$\begin{tikzcd}
\mathcal{HH}(\mathcal{A}_{12^a}) \underset{2}{\circ} \mathcal{HH}(\mathcal{A}_{23^a}) \underset{3}{\circ} \mathcal{HH}(\mathcal{A}_{34^a}) \arrow[r] \arrow[d] & \mathrm{R}\mathcal{H}om(S_{12} \underset{2^2}{\circ} S_{23}, K_{12} \underset{2^2}{\circ} K_{23}) \underset{3}{\circ} \mathcal{HH}(\mathcal{A}_{24^a}) \arrow[d] \\
\mathcal{HH}(\mathcal{A}_{12^a}) \underset{2}{\circ} \mathrm{R}\mathcal{H}om(S_{23} \underset{3^2}{\circ} S_{34}, K_{23} \underset{3^2}{\circ} K_{34}) \arrow[d] & \mathrm{R}\mathcal{H}om((S_{12} \underset{2^2}{\circ} S_{23}) \underset{3^2}{\circ} S_{34}, (K_{12} \underset{2^2}{\circ} K_{23}) \underset{3^2}{\circ} K_{34}) \arrow[d] \\
\mathrm{R}\mathcal{H}om(S_{12} \underset{2^2}{\circ} (S_{23} \underset{3^2}{\circ} S_{34}), K_{12} \underset{2^2}{\circ} (K_{23} \underset{3^2}{\circ} K_{34})) \arrow[r] & \mathrm{R}\mathcal{H}om((S_{12}S_{23}S_{34}) \underset{2^43^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a}), (K_{12}K_{23}K_{34}) \underset{2^43^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a})).
\end{tikzcd} \tag{3.3.8}$$

Following the proof of [KS12, Proposition 4.2.1], we have a morphism

$$K_{ij} \underset{j^2}{\circ} K_{jk} \to K_{ik} \tag{3.3.9}$$

constructed as follows

$$\begin{aligned}
(\mathcal{C}_i\omega_{j^a}) \overset{L}{\underset{\mathcal{A}_{jj^a}}{\otimes}} (\mathcal{C}_j\omega_{k^a}) &\simeq ((\mathcal{C}_i\omega_{j^a})(\mathcal{C}_j\omega_{k^a})) \overset{L}{\underset{\mathcal{A}_{jj^a(jj^a)^a}}{\otimes}} \mathcal{C}_{jj^a} \\
&\simeq ((\mathcal{C}_i\omega_{k^a})(\omega_{j^a}\mathcal{C}_j)) \overset{L}{\underset{\mathcal{A}_{jj^a(jj^a)^a}}{\otimes}} \mathcal{C}_{jj^a} \\
&\simeq (\mathcal{C}_i\omega_{k^a}\omega_{j^a}) \overset{L}{\underset{\mathcal{A}_{jj^a}}{\otimes}} \mathcal{C}_j \\
&\to [(\mathcal{C}_i\omega_k)p_j^{-1}\delta_{j*}\Omega_j^{\mathcal{A}}] \overset{L}{\underset{\mathcal{D}_j^{\mathcal{A}}}{\otimes}} p_j^{-1}\delta_{*j}\mathcal{A}_j \xleftarrow{\sim} p_{ik}^{-1}(\mathcal{C}_i\omega_k)[2d_j].
\end{aligned}$$

where $\mathcal{D}_j^{\mathcal{A}}$ is the quantized ring of differential operator with respect to $\mathcal{A}_j$ (see Definition 2.5.1 of [KS12]) and $\Omega_j^{\mathcal{A}}$ is the quantized module of differential form with respects to $\mathcal{A}_j$ (see Definition 2.5.5 of [KS12]). By [KS12, Lemma 2.5.5] there is an isomorphism $\Omega_j^{\mathcal{A}} \overset{\mathrm{L}}{\underset{\mathcal{D}_j^{\mathcal{A}}}{\otimes}} \mathcal{A}_j[-d_j] \simeq \mathbb{C}_j^{\hbar}$ where d_j denotes the complex dimension of X_j. This isomorphism gives the last arrow in the construction of morphism (3.3.9).
By adjunction between $Rp_{ik!}$ and $p_{ik}^! \simeq p_{ik}^{-1}[2d_j]$, we get the morphism (3.3.9). Choosing $i = 1$, $j = 23$ and $k = 4$, we get the morphism

$$(\mathcal{C}_1\omega_{4^a}\omega_{2^a3^a}) \underset{2^23^2}{\circ} \mathcal{C}_{23} \to \mathcal{C}_1\omega_{4^a}.$$

There are the isomorphisms

$$\begin{aligned}(K_{12}K_{23}K_{34}) \underset{2^43^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a}) &\simeq ((\mathcal{C}_1\omega_{4^a}\omega_{2^a3^a})\mathcal{C}_{23}) \underset{2^43^4}{\circ} (\mathcal{C}_{232^a3^a}) \\ &\simeq (\mathcal{C}_1\omega_{4^a}\omega_{2^a3^a}) \underset{2^23^2}{\circ} \mathcal{C}_{23}.\end{aligned}$$

Thus, we get a map

$$(K_{12}K_{23}K_{34}) \underset{2^43^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a}) \to K_{14}.$$

By construction of the morphism (3.3.9) and of the isomorphism of Proposition 3.3.9 (ii), the below diagram commutes

$$\begin{tikzcd}
(K_{12} \underset{2^2}{\circ} K_{23}) \underset{3^2}{\circ} K_{34} \arrow[r] & K_{13} \underset{3^2}{\circ} K_{34} \arrow[d] \\
(K_{12}K_{23}K_{34}) \underset{2^43^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a}) \arrow[r] \arrow[u, "\wr"'] \arrow[d, "\wr"] & K_{14} \\
K_{12} \underset{2^2}{\circ} (K_{23} \underset{3^2}{\circ} K_{34}) \arrow[r] & K_{12} \underset{2^2}{\circ} K_{24}. \arrow[u]
\end{tikzcd} \tag{3.3.10}$$

Similarly, we get the following commutative diagram

$$\begin{tikzcd}
S_{13} \underset{3^2}{\circ} S_{34} \arrow[r] & S_{12} \underset{2^2}{\circ} (S_{23} \underset{3^2}{\circ} S_{34}) \\
S_{14} \arrow[r] \arrow[u] \arrow[d] & (S_{12}S_{23}S_{34}) \underset{2^43^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a}) \arrow[u, "\wr"'] \arrow[d, "\wr"] \\
S_{12} \underset{2^2}{\circ} S_{24} \arrow[r] & (S_{12} \underset{2^2}{\circ} S_{23}) \underset{3^2}{\circ} S_{34}.
\end{tikzcd} \tag{3.3.11}$$

It follows from the commutation of the diagrams (3.3.10) and (3.3.11) that the diagram below commutes.

$$\begin{tikzcd}
\mathrm{R}\mathcal{H}om(S_{12} \underset{2^2}{\circ} S_{23}, K_{12} \underset{2^2}{\circ} K_{23}) \underset{3}{\circ} \mathcal{HH}(\mathcal{A}_{34^a}) \arrow[r] \arrow[d] & \mathcal{HH}(\mathcal{A}_{13^a}) \underset{3}{\circ} \mathcal{HH}(\mathcal{A}_{34^a}) \arrow[d] \\
\mathrm{R}\mathcal{H}om((S_{12} \underset{2^2}{\circ} S_{23}) \underset{3^2}{\circ} S_{34}, (K_{12} \underset{2^2}{\circ} K_{23}) \underset{3^2}{\circ} K_{34}) \arrow[r] & \mathrm{R}\mathcal{H}om(S_{13} \underset{3^2}{\circ} S_{34}, K_{13} \underset{3^2}{\circ} K_{34}) \arrow[d] \\
\mathrm{R}\mathcal{H}om((S_{12}S_{23}S_{34}) \underset{2^4 3^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a}), (K_{12}K_{23}K_{34}) \underset{2^4 3^4}{\circ} (\mathcal{C}_{22^a}\mathcal{C}_{33^a})) \arrow[r] \arrow[u, "\wr"] \arrow[d, "\wr"'] & \mathcal{HH}(\mathcal{A}_{14}) \\
\mathrm{R}\mathcal{H}om(S_{12} \underset{2^2}{\circ} (S_{23} \underset{3^2}{\circ} S_{34}), K_{12} \underset{2^2}{\circ} (K_{23} \underset{3^2}{\circ} K_{34})) \arrow[r] & \mathrm{R}\mathcal{H}om(S_{12} \underset{2^2}{\circ} S_{24}, K_{12} \underset{2^2}{\circ} K_{24}) \arrow[u] \\
\mathcal{HH}(\mathcal{A}_{12}) \underset{2}{\circ} \mathrm{R}\mathcal{H}om(S_{23} \underset{3^2}{\circ} S_{34}, K_{23} \underset{3^2}{\circ} K_{34}) \arrow[r] \arrow[u] & \mathcal{HH}(\mathcal{A}_{12}) \underset{2}{\circ} \mathcal{HH}(\mathcal{A}_{24}). \arrow[u]
\end{tikzcd} \tag{3.3.12}$$

The commutativity of the diagram (3.3.8) and (3.3.12) prove (i).

(ii) is a consequence of (i) and of Proposition 3.3.4 (ii).

□

3.3.3 Hochschild class

Let $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$. We have the chain of morphisms

$$\begin{aligned}
\mathrm{hh}_{\mathcal{M}} : \mathrm{R}\mathcal{H}om_{\mathcal{A}_X}(\mathcal{M}, \mathcal{M}) &\xleftarrow{\sim} \mathbb{D}'_{\mathcal{A}_X}(\mathcal{M}) \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M} \\
&\simeq \delta^{-1}(\mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} (\mathcal{M} \overset{\mathrm{L}}{\boxtimes} \mathbb{D}'_{\mathcal{A}_X}(\mathcal{M}))) \\
&\to \delta^{-1}(\mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} \mathcal{C}_X).
\end{aligned}$$

We get a map

$$\mathrm{hh}^0_{\mathcal{M}} : \mathrm{Hom}_{\mathcal{A}_X}(\mathcal{M}, \mathcal{M}) \to \mathrm{H}^0_{\mathrm{Supp}(\mathcal{M})}(X, \mathcal{HH}(\mathcal{A}_X)). \tag{3.3.13}$$

Definition 3.3.11. The image of an endomorphism f of $\mathcal{M}$ by the map (3.3.13) gives an element $\mathrm{hh}_X(\mathcal{M}, f) \in \mathrm{H}^0_{\mathrm{Supp}(\mathcal{M})}(X, \mathcal{HH}(\mathcal{A}_X))$ called the Hochschild class of the pair $(\mathcal{M}, f)$. If $f = \mathrm{id}_{\mathcal{M}}$, we simply write $\mathrm{hh}_X(\mathcal{M})$ and call it the Hochschild class of $\mathcal{M}$.

Remark 3.3.12. Let $M \in \mathsf{D}^b_f(\mathbb{C}^\hbar)$ and let $f \in \mathrm{Hom}_{\mathbb{C}^\hbar}(M, M)$. Then the Hochschild class $\mathrm{hh}_{\mathbb{C}^\hbar}(M, f)$ of f is obtained by the composition

$$\begin{aligned}
\mathbb{C}^\hbar \to \mathrm{RHom}_{\mathbb{C}^\hbar}(M, M) \to & M \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} \mathrm{RHom}_{\mathbb{C}^\hbar}(M, \mathbb{C}^\hbar) \xrightarrow{f \otimes \mathrm{id}} M \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} \mathrm{RHom}_{\mathbb{C}^\hbar}(M, \mathbb{C}^\hbar) \\
&\to \mathrm{RHom}_{\mathbb{C}^\hbar}(M, \mathbb{C}^\hbar) \overset{\mathrm{L}}{\underset{\mathbb{C}^\hbar}{\otimes}} M \to \mathbb{C}^\hbar.
\end{aligned}$$

Thus, it is the trace of f in $\mathsf{D}^b(\mathbb{C}^\hbar)$.

3.3.4 Actions of Kernels

We explain how kernels act on Hochschild homology. Let X_1 and X_2 be compact complex manifolds endowed with DQ-algebroids $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\lambda \in \mathrm{HH}_0(\mathcal{A}_{12^a})$. There is a morphism

$$\Phi_\lambda : \mathbb{HH}(\mathcal{A}_2) \to \mathbb{HH}(\mathcal{A}_1) \tag{3.3.14}$$

given by

$$\mathbb{HH}(\mathcal{A}_2) \simeq \mathbb{C}^\hbar \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_2) \overset{\lambda\otimes\mathrm{id}}{\to} \mathbb{HH}(\mathcal{A}_{12^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_2) \overset{\underset{2}{\cup}}{\to} \mathbb{HH}(\mathcal{A}_1).$$

If $\mathcal{K}$ is an object of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$ then there is a morphism

$$\Phi_{\mathcal{K}} : \mathbb{HH}(\mathcal{A}_2) \to \mathbb{HH}_(\mathcal{A}_1) \tag{3.3.15}$$

obtained from morphism (3.3.14) by choosing $\lambda = \mathrm{hh}_{X_{12^a}}(\mathcal{K})$. In [KS12], the authors give initially a different definition and show in [KS12, Lemma 4.3.4] that it is equivalent to the present definition.

We denote by ω_X^{top} the dualizing complex of the category $\mathsf{D}^+(\mathbb{C}_X^\hbar)$.

Proposition 3.3.13. *Let X_i, $(i = 1, 2)$ be a compact complex manifold endowed with a DQ-algebroid $\mathcal{A}_i$.*

(i) The following diagram commutes.

$$\begin{tikzcd} p_1^{-1}\mathcal{HH}(\mathcal{A}_{1^a}) \overset{\mathrm{L}}{\otimes} \mathcal{HH}(\mathcal{A}_{12^a}) \overset{\mathrm{L}}{\otimes} p_2^{-1}\mathcal{HH}(\mathcal{A}_2) \arrow[r, "\cdot\underset{1}{\cup}\cdot\underset{2}{\cup}\cdot"] \arrow[d] & \omega_{12}^{\mathrm{top}} \\ \mathcal{HH}(\mathcal{A}_{12^a}) \overset{\mathrm{L}}{\otimes} \mathcal{HH}(\mathcal{A}_{1^a2}) \arrow[ur, "\underset{1^a2}{\cup}"'] & \end{tikzcd} \tag{3.3.16}$$

(ii) The diagram

$$\begin{tikzcd} \mathbb{HH}(\mathcal{A}_{1^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{12^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_2) \arrow[r, "\cdot\underset{1}{\cup}\cdot\underset{2}{\cup}\cdot"] \arrow[d] & \mathbb{C}^\hbar \\ \mathbb{HH}(\mathcal{A}_{12^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{1^a2}) \arrow[ur, "\underset{1^a2}{\cup}"'] & \end{tikzcd} \tag{3.3.17}$$

commutes.

Proof. (i) In view of Remark 3.3.8, only usual tensor products are involved. Thus, it is a consequence of the projection formula and of the associativity of the tensor product.

(ii) follows from (i).

□

The composition

$$\mathbb{C}^\hbar_{X\times X^a} \to \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{C}_{X\times X^a}}(\mathcal{C}_X, \mathcal{C}_X) \overset{\mathrm{hh}_{\mathcal{C}_X}}{\to} \mathcal{HH}(X \times X^a)$$

induces a map

$$\mathrm{hh}(\Delta_X) : \mathbb{C}^\hbar \to \mathbb{HH}(\mathcal{A}_{X\times X^a}). \tag{3.3.18}$$

The image of $1_{\mathbb{C}^\hbar}$ by $\mathrm{hh}(\Delta_X)$ is $\mathrm{hh}_{X\times X^a}(\mathcal{C}_X)$.

Proposition 3.3.14. *The left (resp. right) actions of* $\mathrm{hh}_{X\times X^a}(\mathcal{C}_X)$ *on* $\mathbb{HH}(\mathcal{A}_X)$ *(resp.* $\mathbb{HH}(\mathcal{A}_{X^a})$*) via the morphism (3.3.6) are the trivial action.*

Proof. See [KS12, Lemma 4.3.2]. □

We define the morphism $\zeta : \mathbb{HH}(\mathcal{A}_{X\times X^a}) \to \mathbb{C}^\hbar$ as the composition

$$\mathbb{HH}(\mathcal{A}_{X^a\times X}) \simeq \mathbb{C}^\hbar \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{X^a\times X}) \xrightarrow{\mathrm{hh}(\Delta_X)\otimes\mathrm{id}} \mathbb{HH}(\mathcal{A}_{X\times X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{X^a\times X}) \xrightarrow{\underset{X^a\times X}{\cup}} \mathbb{C}^\hbar.$$

Corollary 3.3.15. *Let X be a compact complex manifold endowed with a DQ-algebroid $\mathcal{A}_X$. The diagram below commutes.*

$$\begin{tikzcd}
\mathbb{HH}(\mathcal{A}_{X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_X) \arrow[r, "\underset{X}{\cup}"] \arrow[d, "\mathfrak{K}"] & \mathbb{C}^\hbar \\
\mathbb{HH}(\mathcal{A}_{X^a\times X}). \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

Proof. It follows from Proposition 3.3.13 with $X_1 = X_2 = X$, that the triangle on the right of the below diagram commutes. The commutativity of the square on the left is tautological.

$$\begin{tikzcd}
\mathbb{HH}(\mathcal{A}_{X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{C}^\hbar \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_X) \arrow[r, "\mathrm{id}\otimes\mathrm{hh}(\Delta_X)\otimes\mathrm{id}"] \arrow[d, "\wr"] & \mathbb{HH}(\mathcal{A}_{X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{X\times X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_X) \arrow[r, "\cdot\underset{X}{\cup}\cdot\underset{X}{\cup}\cdot"] \arrow[d] & \mathbb{C}^\hbar \\
\mathbb{C}^\hbar \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_X) \arrow[r, "\mathrm{hh}(\Delta_X)\otimes\mathfrak{K}"] & \mathbb{HH}(\mathcal{A}_{X\times X^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{X^a\times X}) \arrow[ur, "\underset{X^a\times X}{\cup}"'] &
\end{tikzcd}$$

□

Finally, an important result is the Theorem 4.3.5 of [KS12]:

Theorem 3.3.16. *Let Λ_i be a closed subset of $X_i \times X_{i+1}$ $(i = 1\ 2)$ and assume that $\Lambda_1 \times_{X_2} \Lambda_2$ is proper over $X_1 \times X_3$. Set $\Lambda = \Lambda_1 \circ \Lambda_2$. Let $\mathcal{K}_i \in \mathsf{D}^b_{\mathrm{coh},\Lambda_i}(\mathcal{A}_{X_i\times X^a_{i+1}})(i = 1,\ 2)$. Then*

$$\mathrm{hh}_{X_{13^a}}(\mathcal{K}_1 \circ \mathcal{K}_2) = \mathrm{hh}_{X_{12^a}}(\mathcal{K}_1) \underset{2}{\cup} \mathrm{hh}_{X_{23^a}}(\mathcal{K}_2) \tag{3.3.19}$$

as elements of $\mathrm{HH}^0_\Lambda(\mathcal{A}_{X_1\times X_{3^a}})$.

Proof. See [KS12, p. 111]. □

3.4 A Lefschetz formula for DQ-modules

3.4.1 The monoidal category of DQ-algebroid stacks

In this subsection we collect a few facts concerning the product $\cdot\boxtimes\cdot$ of DQ-algebroids. Recall that if X and Y are two complex manifolds endowed with DQ-algebroids $\mathcal{A}_X$ and $\mathcal{A}_Y$, $X \times Y$ is canonically endowed with the DQ-algebroid $\mathcal{A}_{X\times Y} := \mathcal{A}_X \boxtimes \mathcal{A}_Y$. There is a functorial symmetry isomorphism

$$\sigma_{X,Y} : (X \times Y, \mathcal{A}_{X\times Y}) \xrightarrow{\sim} (Y \times X, \mathcal{A}_{Y\times X})$$

and for any triple $(X, \mathcal{A}_X)$, $(Y, \mathcal{A}_Y)$ and $(Z, \mathcal{A}_Z)$ there is a natural associativity isomorphism

$$\rho_{X,Y,Z} : (\mathcal{A}_X \boxtimes \mathcal{A}_Y) \boxtimes \mathcal{A}_Z \xrightarrow{\sim} \mathcal{A}_X \boxtimes (\mathcal{A}_Y \boxtimes \mathcal{A}_Z).$$

We consider the category $\mathscr{D}\mathscr{Q}$ whose objects are the pairs $(X, \mathcal{A}_X)$ where X is a complex manifold and $\mathcal{A}_X$ a DQ-algebroid stack on X and where the morphisms are obtained by composing and tensoring the identity morphisms, the symmetry morphisms and the associativity morphisms. The category $\mathscr{D}\mathscr{Q}$ endowed with $\boxtimes$ is a symmetric monoidal category.

We denote by

$$v : ((X \times Y) \times (X \times Y)^a, \mathcal{A}_{(X\times Y)\times(X\times Y)^a}) \to ((Y \times X) \times (Y \times X)^a, \mathcal{A}_{(Y\times X)\times(Y\times X)^a})$$

the map defined by $v := \sigma \times \sigma$.

In this situation, after identifying, $(X \times X^a) \times (Y \times Y^a)$ with $(X \times Y) \times (X \times Y)^a$, there is a natural isomorphism $\mathcal{C}_X \overset{\mathrm{L}}{\boxtimes} \mathcal{C}_Y \simeq \mathcal{C}_{X\times Y}$ and the morphism v induces an isomorphism

$$v_*(\mathcal{C}_{X\times Y}) \simeq \mathcal{C}_{Y\times X}.$$

Proposition 3.4.1. *The map $\sigma_{X,Y}$ induce an isomorphism*

$$\sigma_* : \sigma_{X,Y*}(\mathcal{HH}(\mathcal{A}_{X\times Y})) \to \mathcal{HH}(\mathcal{A}_{Y\times X}) \tag{3.4.1}$$

Proof. There is the following Cartesian square of topological space.

$$\begin{tikzcd} X \times Y \arrow[r, "\sigma"] \arrow[d, "\delta_{X\times Y}"] & Y \times X \arrow[d, "\delta_{Y\times X}"] \\ (X \times Y) \times (X \times Y) \arrow[r, "v"] & (Y \times X) \times (Y \times X). \end{tikzcd}$$

Then

$$\begin{aligned} \sigma_* \mathcal{HH}(\mathcal{A}_{X\times Y}) &\simeq \sigma_! \delta^{-1}_{X\times Y}(\mathcal{C}_{(X\times Y)^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X\times Y}}{\otimes}} \mathcal{C}_{X\times Y}) \\ &\simeq \delta^{-1}_{Y\times X} v_!(\mathcal{C}_{(X\times Y)^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X\times Y}}{\otimes}} \mathcal{C}_{X\times Y}) \\ &\simeq \delta^{-1}_{Y\times X}(\mathcal{C}_{(Y\times X)^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{Y\times X}}{\otimes}} \mathcal{C}_{Y\times X}). \end{aligned}$$

□

The morphsim (3.4.1) induces an isomorphism that we still denote σ_*

$$\sigma_* : \mathbb{HH}(\mathcal{A}_{X\times Y}) \to \mathbb{HH}(\mathcal{A}_{Y\times X}).$$

The following diagram commutes

$$\begin{tikzcd} \mathbb{HH}(\mathcal{A}_{X\times Y}) \arrow[r, "\sigma_*"] & \mathbb{HH}(\mathcal{A}_{Y\times X}) \\ \mathbb{HH}(\mathcal{A}_X) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_Y) \arrow[u, "\mathfrak{K}"] \arrow[r] & \mathbb{HH}(\mathcal{A}_Y) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_X). \arrow[u, "\mathfrak{K}"'] \end{tikzcd} \tag{3.4.2}$$

Proposition 3.4.2. *There is the equality*

$$\sigma_* \operatorname{hh}_{X \times X^a}(\mathcal{C}_X) = \operatorname{hh}_{X^a \times X}(\mathcal{C}_{X^a}).$$

Proof. Immediate by using Lemma 4.1.4 of [KS12]. $\square$

Proposition 3.4.3. *Let X and Y be compact complex manifolds endowed with DQ-algebroids $\mathcal{A}_X$ and $\mathcal{A}_Y$. The diagram*

$$\begin{tikzcd}
\mathbb{HH}(\mathcal{A}_{(Y \times X)^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{Y \times X}) \arrow[r, "\underset{Y \times X}{\cup}"] & \mathbb{C}^{\hbar} \\
\mathbb{HH}(\mathcal{A}_{(X \times Y)^a}) \overset{\mathrm{L}}{\otimes} \mathbb{HH}(\mathcal{A}_{X \times Y}) \arrow[u, "\sigma_* \otimes \sigma_*"] \arrow[ur, "\underset{X \times Y}{\cup}"'] &
\end{tikzcd}$$

commutes

Proof. It is sufficient to show that the below diagram is commutative.

$$\begin{tikzcd}
\mathcal{HH}(\mathcal{A}_{(Y \times X)^a}) \underset{\mathbb{C}_{Y \times X}}{\overset{\mathrm{L}}{\otimes}} \mathcal{HH}(\mathcal{A}_{Y \times X}) \arrow[r, "\underset{Y \times X}{\cup}"] & \omega^{top}_{Y \times X} \\
v_*(\mathcal{HH}(\mathcal{A}_{(X \times Y)^a}) \underset{\mathbb{C}_{X \times Y}}{\overset{\mathrm{L}}{\otimes}} \mathcal{HH}(\mathcal{A}_{X \times Y})) \arrow[u, "v_*"] \arrow[ur, "\underset{X \times Y}{\cup}"'] &
\end{tikzcd}$$

For that purpose, we have to go through the construction of $\cup$ on page 104 of [KS12]. Let us make some observations.

Since X and Y are compact $\sigma_* = \sigma_!$. By the base change formula we have $\sigma_! \delta_{X \times Y} \simeq \delta_{Y \times X} v_!$.

The morphism v is a closed immersion. Thus, setting $Z = X \times Y$ and $Z' = Y \times X$, we have the isomorphism $v_* \mathrm{R}\mathcal{H}om_{\mathcal{A}_{Z \times Z}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{A}_{Z' \times Z'}}(v_* \mathcal{M}, v_* \mathcal{N})$ for every $\mathcal{M}$, $\mathcal{N} \in \operatorname{Mod}(\mathcal{A}_{Z \times Z})$.

In our case, we apply the construction of the pairing with $X_1 = X_3 = \mathrm{pt}$ and $X_2 = X \times Y$. Hence by Remark 3.3.8, we can replace the completed tensor products appearing in the construction of the pairing by usual tensor products and use the isomorphism $v_*(\mathcal{M} \underset{\mathcal{A}_{Z \times Z}}{\overset{\mathrm{L}}{\otimes}} \mathcal{N}) \simeq v_* \mathcal{M} \underset{\mathcal{A}_{Z' \times Z'}}{\overset{\mathrm{L}}{\otimes}} v_* \mathcal{N}$ for every $\mathcal{M}, \mathcal{N} \in \operatorname{Mod}(\mathcal{A}_{Z \times Z})$.

We also notice that $v_* \mathcal{C}_Z \simeq \mathcal{C}_{Z'}$, $v_* \omega_Z \simeq \omega_{Z'}$ and $v_* \omega_Z^{-1} \simeq \omega_{Z'}^{-1}$. Similar results holds when replacing Z by Z^a and Z' by $Z^{a\prime}$. The result follows easily from these observations. $\square$

3.4.2 The Lefschetz-Lunts formula for DQ-modules

Inspired by the Lefschetz formula for Fourier-Mukai functor of V. Lunts (see [Lun11]), we give a similar formula in the framework of DQ-modules.

Theorem 3.4.4. *Let X be a compact complex manifold equiped with a DQ-algebroid $\mathcal{A}_X$. Let $\lambda \in \mathrm{HH}_0(\mathcal{A}_{X \times X^a})$. Consider the map (3.3.14)*

$$\Phi_\lambda : \mathbb{HH}(\mathcal{A}_X) \to \mathbb{HH}(\mathcal{A}_X).$$

Then

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\lambda) = \mathrm{hh}_{X^a \times X}(\mathcal{C}_{X^a}) \underset{X \times X^a}{\cup} \lambda.$$

Proof. Consider the full subcategory $\mathscr{C}$ of $\mathscr{DQ}$ whose objects are the pair $(X, \mathcal{A}_X)$ where X is a compact manifold. By the results of Subsection 3.4.1, the pair $(\mathbb{HH}, \mathfrak{K})$ is a symmetric monoidal functor.

The data are given by

(a) the functor $(\cdot)^a$ which associate to a DQ-algebroid $(X, \mathcal{A}_X)$ the opposite DQ-algebroid $(X, \mathcal{A}_{X^a})$,

(b) the monoidal functor on $\mathscr{C}$ given by the pair $(\mathbb{HH}, \mathfrak{K})$,

(c) the morphism (3.3.6),

(d) for each pair $(X, \mathcal{A}_X)$ the morphism $\mathrm{hh}(\Delta_X)$.

We check the properties requested by of our formalism:

(i) the Property (P1) is granted by Corollary 3.3.6,

(ii) the Property (P2) follows from Proposition 3.3.10,

(iii) the Property (P3) follows from Proposition 3.4.2,

(iv) the Property (P4) follows from Proposition 3.3.14,

(v) the Property (P5) follows from Proposition 3.3.15,

(vi) the Property (P6) follows from Proposition 3.4.3.

Then the formula follows from Theorem 3.2.11. □

Corollary 3.4.5. *Let X be a compact complex manifold endowed with a DQ-algebroid $\mathcal{A}_X$ and let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X \times X^a})$. Then*

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\mathcal{K}) = \mathrm{hh}_{X^a \times X}(\mathcal{C}_{X^a}) \underset{X \times X^a}{\cup} \mathrm{hh}_{X \times X^a}(\mathcal{K}).$$

Proof. Apply Theorem 3.4.4 to $\Phi_\mathcal{K}$. □

Corollary 3.4.6. *Let X be a compact complex manifold endowed with a DQ-algebroid $\mathcal{A}_X$ and let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X \times X^a})$. Then*

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\mathcal{K}) = \chi(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} \mathcal{K})).$$

Proof. By Corollary 3.4.5, we get that

$$\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\mathcal{K}) = \mathrm{hh}_{X^a \times X}(\mathcal{C}_{X^a}) \underset{X \times X^a}{\cup} \mathrm{hh}_{X \times X^a}(\mathcal{K}).$$

Applying Theorem 3.3.16 with $X_1 = X_3 = \mathrm{pt}$ and $X_2 = X \times X^a$ we find that

$$\mathrm{hh}_{X^a \times X}(\mathcal{C}_{X^a}) \underset{X \times X^a}{\cup} \mathrm{hh}_{X \times X^a}(\mathcal{K}) = \mathrm{hh}_{\mathrm{pt}}(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} \mathcal{K}).$$

By Remark 3.3.12, it follows that

$$\mathrm{hh}_{\mathrm{pt}}(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} \mathcal{K}) = \chi(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} \mathcal{K})).$$

Finally, we get that $\mathrm{Tr}_{\mathbb{C}^\hbar}(\Phi_\mathcal{K}) = \chi(\mathrm{R}\Gamma(X \times X^a; \mathcal{C}_{X^a} \overset{\mathrm{L}}{\underset{\mathcal{A}_{X \times X^a}}{\otimes}} \mathcal{K}))$. □

3.4.3 Applications

We give some consequences of Theorem 3.4.4 and explain how to recover some of the results of the paper [Lun11] of V. Lunts and give a special form of the formula when X is also symplectic.

Theorem 3.4.7 ([Lun11]). *Let X be a compact complex manifold and $\mathcal{K}$ an object of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_{X\times X})$. Then,*

$$\sum_i (-1)^i \operatorname{Tr}(\mathrm{H}^i(\Phi_{\mathcal{K}})) = \chi(\mathrm{R}\Gamma(X\times X; \mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{O}_{X\times X}}{\otimes}} \mathcal{K})).$$

Proof. We endow X with the trivial deformation. Then, we can apply Corollary 3.4.6 and forget $\hbar$ by applying $\mathrm{gr}_\hbar$. We recover Theorem 3.9 of [Lun11]. □

Proposition 3.4.8. *Let X be a compact complex manifold endowed with a DQ-algebroid $\mathcal{A}_X$ and let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X\times X^a})$. Then*

$$\operatorname{Tr}(\Phi_{\mathcal{K}}) = \operatorname{Tr}(\Phi_{\mathrm{gr}_\hbar \mathcal{K}}).$$

Proof. Remark that

$$\chi(\mathrm{RHom}_{\mathcal{A}_X}(\omega_X^{-1}, \mathcal{K})) = \chi(\mathrm{RHom}_{\mathrm{gr}_\hbar \mathcal{A}_X}((\mathrm{gr}_\hbar \omega_X^{-1}), \mathrm{gr}_\hbar \mathcal{K})).$$

Then, the result follows by Corollary 3.4.6 and Theorem 3.4.7. □

It is possible to localize $\mathcal{A}_X$ with respect to $\hbar$. We denote by $\mathbb{C}((\hbar))$ the field of formal Laurent series. We set $\mathcal{A}_X^{loc} = \mathbb{C}((\hbar)) \otimes \mathcal{A}_X$. If $\mathcal{M}$ is a $\mathcal{A}_X$-module we denote by $\mathcal{M}^{loc}$ the $\mathcal{A}_X^{loc}$-module $\mathbb{C}((\hbar)) \otimes \mathcal{M}$.

Corollary 3.4.9. *Let X be a compact complex manifold endowed with a DQ-algebroid $\mathcal{A}_X$ and let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X\times X^a})$. Then,*

$$\sum_i (-1)^i \operatorname{Tr}(\mathrm{H}^i(\Phi_{\mathcal{K}})) = \int_X \delta^* \operatorname{ch}(\mathrm{gr}_\hbar \mathcal{K}) \cup \operatorname{td}_X(TX)$$

where $\operatorname{ch}(\mathrm{gr}_\hbar \mathcal{K})$ is the Chern class of $\mathrm{gr}_\hbar \mathcal{K}$, $\operatorname{td}_X(TX)$ is the Todd class of the tangent bundle TX and δ^* is the pullback by the diagonal embedding.

Proof. By Corollary 3.4.6, we have $\operatorname{Tr}(\Phi_{\mathcal{K}}) = \chi(\mathrm{RHom}_{\mathcal{A}_X}(\omega_X^{-1}, \mathcal{K}))$ and

$$\chi(\mathrm{RHom}_{\mathcal{A}_X}(\omega_X^{-1}, \mathcal{K})) = \chi(\mathrm{RHom}_{\mathcal{A}_X^{loc}}((\omega_X^{-1})^{loc}, \mathcal{K}^{loc})).$$

By Corollary 5.3.5 of [KS12], we have

$$\chi(\mathrm{RHom}_{\mathcal{A}_X^{loc}}((\omega_X^{-1})^{loc}, \mathcal{K}^{loc})) = \textstyle\int_{X\times X} \operatorname{ch}(\delta_* \mathcal{O}_X) \cup \operatorname{ch}(\mathrm{gr}_\hbar \mathcal{K}) \cup \operatorname{td}_{X\times X}(T(X\times X)).$$

Applying the Grothendieck-Riemann-Roch theorem, we have

$$\begin{aligned}\sum_i (-1)^i \operatorname{Tr}(\mathrm{H}^i(\Phi_{\mathcal{K}})) &= \int_X \operatorname{ch}(\mathrm{gr}_\hbar \mathcal{K}) \cup \delta_* \operatorname{td}_X(TX) \\ &= \int_X \delta^* \operatorname{ch}(\mathrm{gr}_\hbar \mathcal{K}) \cup \operatorname{td}_X(TX).\end{aligned}$$

□

We denote by d_X the complex dimension of X. In the symplectic case, we have according to [KS12, §6.3]

Theorem 3.4.10. *If X is a complex symplectic manifold, the complex $\mathcal{HH}(\mathcal{A}_X^{loc})$ is concentrated in degree $-d_X$ and there is a canonical isomorphism*

$$\tau_X : \mathcal{HH}(\mathcal{A}_X^{loc}) \underset{\tau_X}{\xrightarrow{\sim}} \mathbb{C}_X^{\hbar,loc}[d_X].$$

We refer the reader to section 6.2 and 6.3 of [KS12] for a precise description of τ_X. According to [KS12, Definition 6.3.2], the Euler class of a $\mathcal{A}_X^{loc}$-module is defined by

Definition 3.4.11. Let $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X^{loc})$. We set

$$\mathrm{eu}(\mathcal{M}) = \tau_X(\mathrm{hh}_X(\mathcal{M})) \in \mathrm{H}^{d_X}_{\mathrm{Supp}(\mathcal{M})}(X; \mathbb{C}_X)$$

and call $\mathrm{eu}_X(\mathcal{M})$ the Euler class of $\mathcal{M}$.

Therefore, we have the following

Proposition 3.4.12. *Let X be a compact complex symplectic manifold and let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{X \times X^a})$. Then,*

$$\sum_i (-1)^i \operatorname{Tr}(\mathrm{H}^i(\Phi_{\mathcal{K}})) = \int_{X \times X} \mathrm{eu}(\mathcal{C}_X^{loc}) \cup \mathrm{eu}(\mathcal{K}^{loc})$$

where $\cup$ is the cup product.

Proof. It is a direct consequence of [KS12, §6.3] and of Theorem 3.4.4. □

Remark 3.4.13. Similarly, it is possible to apply the results of Section 3.2 to the case of dg algebras to recover the Lefschetz-Lunts formula for dg modules.

Chapter 4

Fourier-Mukai transforms for DQ-modules

4.1 Introduction

Fourier-Mukai transforms have been extensively studied in algebraic geometry. A key results of Orlov (see [Orl97]) states that any equivalence of triangulated categories between the bounded derived categories of coherent sheaves on smooth projective varieties is isomorphic to a Fourier-Mukai functor. We refer the reader to the seminal book of Huybrechts [Huy06] for an in depth treatment of Fourier-Mukai transforms in algebraic geometry. In [KS12], Kashiwara and Schapira have developed a framework to study integral transforms in the setting of DQ-modules. Recently, Arinkin, Block and Pantev have proved in [ABP11] that under some condition an integral transform between complex manifolds can be lifted to the non-commutative level.

The aim of this short chapter is to prove Theorem 4.3.5 which says that for any two smooth complex projective varieties X_1, X_2 equipped respectively with DQ-algebroid stacks $\mathcal{A}_1, \mathcal{A}_2$ and any kernel $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$, the integral transforms $\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ associated to $\mathcal{K}$ is is fully faithful (resp. an equivalence of triangulated categories) if and only if $\Phi_{\mathrm{gr}_\hbar \mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_1)$ is fully faithful (resp. an equivalence of triangulated categories).

We borrow the notation of Chapter 3, especially Notation 3.3.1.

4.2 Duality for DQ-modules

Let X_i $(i = 1, 2, 3)$ be smooth projective complex varieties endowed with the Zariski topology and let $\mathcal{A}_i$ be a DQ-algebroid on X_i. We recall some duality results for DQ-modules from [KS12, Chap. 3].

First we need the following result.

Proposition 4.2.1 ([KS12, p. 93]). *Let $\mathcal{K}_i \in \mathsf{D}^b(\mathcal{A}_{i(i+1)^a})$ $(i = 1, 2)$ and let $\mathcal{L}$ be a bi-invertible $\mathcal{A}_2 \otimes \mathcal{A}_{2^a}$-module. Then, there is a natural isomorphism*

$$(\mathcal{K}_1 \underset{2}{\circ} \mathcal{L}) \underset{2}{\circ} \mathcal{K}_2 \simeq \mathcal{K}_1 \underset{2}{\circ} (\mathcal{L} \underset{2}{\circ} \mathcal{K}_2).$$

We denote by ω_i the dualizing complexe for $\mathcal{A}_i$. It is a bi-invertible $(\mathcal{A}_i \otimes \mathcal{A}_{i^a})$-module. Since the category of bi-invertible $(\mathcal{A}_i \otimes \mathcal{A}_{i^a})$-modules is equivalent to the category of coherent $\mathcal{A}_{ii^a}$-modules simple along the diagonal, we will regard ω_i as an $\mathcal{A}_{ii^a}$-module supported by the diagonal and we will still denote it by ω_i.

Theorem 4.2.2 ([KS12, Theorem 3.3.3]). *Let* $\mathcal{K}_i \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{i(i+1)^a})$ $(i = 1, 2)$. *There is a natural isomorphism in* $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{1^a3})$

$$(\mathbb{D}'_{\mathcal{A}_{12^a}}\mathcal{K}_1) \underset{2^a}{\circ} \omega_{2^a} \underset{2^a}{\circ} (\mathbb{D}'_{\mathcal{A}_{23^a}}\mathcal{K}_2) \xrightarrow{\sim} \mathbb{D}'_{\mathcal{A}_{13^a}}(\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2).$$

4.3 Fourier-Mukai functors in the quantized setting

Following [KS12], we define Fourier-Mukai functors in the framework of DQ-modules. For a closed subset Λ of X , we denote by $\mathsf{D}^b_{\mathrm{coh},\Lambda}(\mathcal{A}_X)$ the full triangulated subscategory of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_X)$ consiting of objects supported by Λ. Recall the following theorem.

Theorem 4.3.1 (Theorem 3.2.1 of [KS12]). *For* $i = 1, 2$, *let* Λ_i *be a closed subset of* $X_i \times X_{i+1}$ *and* $\mathcal{K}_i \in \mathsf{D}^b_{\mathrm{coh},\Lambda_i}(\mathcal{A}_{i(i+1)^a})$. *Assume that* $\Lambda_1 \times_{X_2} \Lambda_2$ *is proper over* $X_1 \times X_3$, *and set* $\Lambda = p_{13}(p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2)$. *Then the object* $\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2$ *belongs to* $\mathsf{D}^b_{\mathrm{coh},\Lambda}(\mathcal{A}_{13^a})$.

From now on, all the varieties considered are smooth complex projective varieties endowed with the Zariski topology. Let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$. Theorem 4.3.1 implies that the functor (4.3.1) is well-defined.

$$\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1), \qquad \mathcal{M} \mapsto \mathcal{K} \underset{2}{\circ} \mathcal{M} = Rp_{1*}(K \overset{\mathrm{L}}{\underset{p_2^{-1}\mathcal{A}_2}{\otimes}} p_2^{-1}\mathcal{M}). \tag{4.3.1}$$

Proposition 4.3.2. *Let* $\mathcal{K}_1 \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$ *and* $\mathcal{K}_2 \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{23^a})$. *The composition*

$$\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_3) \xrightarrow{\Phi_{\mathcal{K}_2}} \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \xrightarrow{\Phi_{\mathcal{K}_1}} \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$$

is isomorphic to $\Phi_{\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_3) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$.

Proof. It is a direct consequence of Proposition 3.3.9. □

Definition 4.3.3. For any object $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$, we set

$$\mathcal{K}_R = \mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{2^a}{\circ} \omega_{2^a} \qquad\qquad \mathcal{K}_L = \mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{1}{\circ} \omega_1$$

objects of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{1^a2})$.

Proposition 4.3.4. *Let* $\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ *be the Fourier-Mukai functor associated to* $\mathcal{K}$ *and* $\Phi_{\mathcal{K}_R} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$ *(resp.* $\Phi_{\mathcal{K}_L} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$*) the Fourier-Mukai functor associated to* $\mathcal{K}_R$ *(resp.* $\mathcal{K}_L$*). Then* $\Phi_{\mathcal{K}_R}$ *(resp.* $\Phi_{\mathcal{K}_L}$*) is right (resp. left) adjoint to* $\Phi_{\mathcal{K}}$.

Proof. We have

$$\mathrm{RHom}_{\mathcal{A}_1}(\mathcal{K} \underset{2}{\circ} \mathcal{M}, \mathcal{N}) \simeq \mathrm{R}\Gamma(X_1, \mathrm{R}\mathcal{H}om_{\mathcal{A}_1}(\mathcal{K} \underset{2}{\circ} \mathcal{M}, \mathcal{N})).$$

Applying Theorem 4.2.2 and the projection formula, we get

$$\begin{aligned}
\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_1}(\mathcal{K} \underset{2}{\circ} \mathcal{M}, \mathcal{N}) &\simeq \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_1}(\mathcal{K} \underset{2}{\circ} \mathcal{M}, \mathcal{A}_1) \overset{\mathrm{L}}{\underset{\mathcal{A}_1}{\otimes}} \mathcal{N} \\
&\simeq (\mathbb{D}'_{\mathcal{A}_{12^a}}(\mathcal{K}) \underset{2^a}{\circ} \omega_{2^a} \underset{2^a}{\circ} \mathbb{D}'_{\mathcal{A}_2}(\mathcal{M})) \overset{\mathrm{L}}{\underset{\mathcal{A}_1}{\otimes}} \mathcal{N} \\
&\simeq (\mathcal{K}_R \underset{2^a}{\circ} \mathbb{D}'_{\mathcal{A}_2}(\mathcal{M})) \overset{\mathrm{L}}{\underset{\mathcal{A}_1}{\otimes}} \mathcal{N} \\
&\simeq \mathrm{R}\, p_{1*}(\mathcal{K}_R \overset{\mathrm{L}}{\underset{p_2^{-1}\mathcal{A}_{2^a}}{\otimes}} p_2^{-1}\mathbb{D}'_{\mathcal{A}_2}(\mathcal{M})) \overset{\mathrm{L}}{\underset{\mathcal{A}_1}{\otimes}} \mathcal{N} \\
&\simeq \mathrm{R}\, p_{1*}(\mathcal{K}_R \overset{\mathrm{L}}{\underset{p_2^{-1}\mathcal{A}_{2^a}}{\otimes}} p_2^{-1}\mathbb{D}'_{\mathcal{A}_2}(\mathcal{M}) \overset{\mathrm{L}}{\underset{p_1^{-1}\mathcal{A}_1}{\otimes}} p_1^{-1}\mathcal{N}).
\end{aligned}$$

Taking the global section and applying again the projection formula, we get

$$\begin{aligned}
\mathrm{R}\Gamma(X_1, \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_1}(\mathcal{K} \underset{2}{\circ} \mathcal{M}, \mathcal{N})) &\simeq \mathrm{R}\Gamma(X_1, \mathrm{R}\, p_{1*}(\mathcal{K}_R \overset{\mathrm{L}}{\underset{p_2^{-1}\mathcal{A}_{2^a}}{\otimes}} p_2^{-1}\mathbb{D}'_{\mathcal{A}_2}(\mathcal{M})) \overset{\mathrm{L}}{\underset{p_1^{-1}\mathcal{A}_1}{\otimes}} p_1^{-1}\mathcal{N}) \\
&\simeq \mathrm{R}\Gamma(X_1 \times X_2, (\mathcal{K}_R \overset{\mathrm{L}}{\underset{p_2^{-1}\mathcal{A}_{2^a}}{\otimes}} p_2^{-1}\mathbb{D}'_{\mathcal{A}_2}(\mathcal{M})) \overset{\mathrm{L}}{\underset{p_1^{-1}\mathcal{A}_1}{\otimes}} p_1^{-1}\mathcal{N}) \\
&\simeq \mathrm{R}\Gamma(X_2, \mathrm{R}\, p_{2*}((\mathcal{K}_R \overset{\mathrm{L}}{\underset{p_2^{-1}\mathcal{A}_{2^a}}{\otimes}} p_2^{-1}\mathbb{D}'_{\mathcal{A}_2}(\mathcal{M})) \overset{\mathrm{L}}{\underset{p_1^{-1}\mathcal{A}_1}{\otimes}} p_1^{-1}\mathcal{N})) \\
&\simeq \mathrm{R}\Gamma(X_2, \mathbb{D}'_{\mathcal{A}_2}(\mathcal{M}) \overset{\mathrm{L}}{\underset{\mathcal{A}_2}{\otimes}} (\mathcal{K}_R \underset{1}{\circ} \mathcal{N})) \\
&\simeq \mathrm{RHom}_{\mathcal{A}_2}(\mathcal{M}, \mathcal{K}_R \underset{1}{\circ} \mathcal{N}).
\end{aligned}$$

Thus, $\mathrm{RHom}_{\mathcal{A}_1}(\mathcal{K} \underset{2}{\circ} \mathcal{M}, \mathcal{N}) \simeq \mathrm{RHom}_{\mathcal{A}_2}(\mathcal{M}, \mathcal{K}_R \underset{1}{\circ} \mathcal{N})$ which proves the claim. The proof is similar for $\mathcal{K}_L$. □

Theorem 4.3.5. *Let X_1 (resp. X_2) be a smooth complex projective variety endowed with a DQ-algebroid $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Let $\mathcal{K} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_{12^a})$. The following conditions are equivalent*

(i) The functor $\Phi_{\mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_1)$ is fully faithful (resp. an equivalence of triangulated categories).

(ii) The functor $\Phi_{\mathrm{gr}_\hbar \mathcal{K}} : \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2) \to \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_1)$ is fully faithful (resp. an equivalence of triangulated categories).

Proof. We recall the following fact. Let F and G be two functors and assume that F is right adjoint to G. Then, there are two natural morphisms

$$G \circ F \to \mathrm{id} \tag{4.3.2}$$

$$\mathrm{id} \to F \circ G. \tag{4.3.3}$$

The morphism (4.3.2) (resp. (4.3.3)) is an isomorphism if and only if F (resp. G) is fully faithfull. The morphism (4.3.2) and (4.3.3) are isomorphisms if and only if F and G are equivalences.

1. $(i) \Rightarrow (ii)$. Proposition 4.3.4 is also true for $\mathcal{O}$-modules since the proof works in the commutative case without any changes. Moreover, the functor $\mathrm{gr}_\hbar$ commutes with the composition of kernels. Hence, we have $\mathrm{gr}_\hbar(\mathcal{K}_R) \simeq (\mathrm{gr}_\hbar \mathcal{K})_R$. Therefore, the

functor $\Phi_{\mathrm{gr}_\hbar \mathcal{K}_R}$ is a right adjoint of the functor $\Phi_{\mathrm{gr}_\hbar \mathcal{K}}$. Thus, there are morphisms of functors

$$\Phi_{\mathrm{gr}_\hbar \mathcal{K}} \circ \Phi_{\mathrm{gr}_\hbar \mathcal{K}_R} \to \mathrm{id}, \tag{4.3.4}$$

$$\mathrm{id} \to \Phi_{\mathrm{gr}_\hbar \mathcal{K}_R} \circ \Phi_{\mathrm{gr}_\hbar \mathcal{K}}. \tag{4.3.5}$$

Set $\Phi_{\mathcal{L}} = \Phi_{\mathrm{gr}_\hbar \mathcal{K}_R} \circ \Phi_{\mathrm{gr}_\hbar \mathcal{K}}$. Let $\mathcal{T}_2$ be the full subcategory of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$ whose objects are the $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$ such that

$$\mathcal{M} \to \Phi_{\mathcal{L}}(\mathcal{M})$$

is an isomorphism. It follows from Proposition 1.3.3 that $\mathcal{T}_2$ is a thick subcategory of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$.
Let $\mathcal{G}$ be a compact generator of $\mathsf{D}_{\mathrm{qcoh}}(\mathcal{O}_2)$. By Lemma 2.3.18, $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2) = \langle \mathcal{G} \rangle$. Since $\Phi_{\mathcal{K}}$ is a fully faithful we have the isomorphism

$$\iota_g(\mathcal{G}) \xrightarrow{\sim} \Phi_{\mathcal{K}_R} \circ \Phi_{\mathcal{K}}(\iota_g(\mathcal{G})).$$

Applying the functor $\mathrm{gr}_\hbar$, we get that $\mathrm{gr}_\hbar\, \iota_g(\mathcal{G})$ belongs to $\mathcal{T}_2$ and by Lemma 2.3.18, $\mathrm{gr}_\hbar\, \iota_g(\mathcal{G})$ is a classical generator of $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$. Hence, $\mathcal{T}_2 = \mathsf{D}^b_{\mathrm{coh}}(\mathcal{O}_2)$. Thus, the morphism (4.3.5) is an isomorphism of functors. A similar argument shows that if $\Phi_{\mathrm{gr}_\hbar \mathcal{K}}$ is an equivalence the morphism (4.3.4) is also an isomorphism which proves the claim.

2. $(ii) \Rightarrow (i)$.
Since $\Phi_{\mathcal{K}}$ and $\Phi_{\mathcal{K}_R}$ are adjoint functors we have natural morphisms of functors

$$\Phi_{\mathcal{K}} \circ \Phi_{\mathcal{K}_R} \to \mathrm{id},$$
$$\mathrm{id} \to \Phi_{\mathcal{K}_R} \circ \Phi_{\mathcal{K}}.$$

If $\mathcal{M} \in \mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$, then we have

$$\mathcal{M} \to \Phi_{\mathcal{K}_R} \circ \Phi_{\mathcal{K}}(\mathcal{M}). \tag{4.3.6}$$

Applying the functor $\mathrm{gr}_\hbar$, we get

$$\mathrm{gr}_\hbar \mathcal{M} \to \Phi_{\mathrm{gr}_\hbar \mathcal{K}_R} \circ \Phi_{\mathrm{gr}_\hbar \mathcal{K}}(\mathrm{gr}_\hbar \mathcal{M}). \tag{4.3.7}$$

If $\Phi_{\mathrm{gr}_\hbar \mathcal{K}}$ is fully faithful, then the morphism (4.3.7) is an isomorphism. The objects $\Phi_{\mathcal{K}_R} \circ \Phi_{\mathcal{K}}(\mathcal{M})$ and $\mathcal{M}$ are cohomologically complete since they belongs to $\mathsf{D}^b_{\mathrm{coh}}(\mathcal{A}_2)$. Thus the morphism (4.3.6) is an isomorphism that is to say

$$\mathrm{id} \xrightarrow{\sim} \Phi_{\mathcal{K}_R} \circ \Phi_{\mathcal{K}}.$$

It follows that $\Phi_{\mathcal{K}}$ is fully faithful.
Similarly, one shows that if $\Phi_{\mathrm{gr}_\hbar \mathcal{K}}$ is an equivalence then in addition

$$\Phi_{\mathcal{K}} \circ \Phi_{\mathcal{K}_R} \xrightarrow{\sim} \mathrm{id}\,.$$

It follows that $\Phi_{\mathcal{K}}$ is an equivalence.

□

Remark 4.3.6. The implication $(ii) \Rightarrow (i)$ of Theorem 4.3.5 and Proposition 4.3.4 still hold if one replaces smooth projective varieties by complex compact manifolds.

Appendix A

Algebroid stacks

The aim of this appendix is to present some notions of the theory of algebroid stacks. They have been introduced by Kontsevich in [Kon01]. They are used in the deformation quantization of complex Poisson varieties. We assume that the reader has some familiarities with the language of stacks. Our presentation follows closely [KS12, §2.1] and [KS06, chapter 19].

In this appendix k is a commutative unital ring and X is a topological space.

A.1 Algebroid stacks

Stacks are roughly speaking sheaves of categories. In some sense algebroid stacks are the counterpart in the language of stacks of the notion of sheaves of rings.

Definition A.1.1. A k-linear category is a category $\mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is endowed with a k-module structure for any X, $Y \in \mathrm{Ob}(\mathscr{C})$ and such that the composition map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$ is a k-bilinear map for any X, Y, $Z \in \mathscr{C}$.

Definition A.1.2. (i) A k-linear stack $\mathfrak{S}$ on X is a stack such that for every open subset U of X, $\mathfrak{S}(U)$ is a k-linear category.

(ii) Let $\mathcal{R}$ be a sheaf of commutative ring on X. An $\mathcal{R}$-linear stack, is a $\mathbb{Z}$-linear stack $\mathfrak{S}$ such that the sheaf $\mathcal{E}nd(\mathrm{id}_{\mathfrak{S}})$ of endomorphism of the identity functor $\mathrm{id}_{\mathfrak{S}}$ from $\mathfrak{S}$ to itself, is a sheaf of commutative $\mathcal{R}$-algebras.

Definition A.1.3. (i) A k-algebroid stack $\mathcal{A}$ on X is a k-linear stack which is locally non-empty (i.e there exists a covering of X by open subsets $\{U_i\}_{i \in I}$ such that for every $i \in I$, the category $\mathcal{A}(U_i)$ is non-empty) and such that for any open subset U of X, two objects of $\mathcal{A}(U)$ are locally isomorphic.

(ii) An $\mathcal{R}$-algebroid stack $\mathcal{A}$ on X is a $\mathbb{Z}$-algebroid stack endowed with a morphism of rings $\mathcal{R} \to \mathcal{E}nd(\mathrm{id}_{\mathfrak{S}})$.

(iii) An $\mathcal{R}$-algebroid $\mathcal{A}$ is called an invertible $\mathcal{R}$-algebroid if $\mathcal{R}|_U \to \mathcal{E}nd_{\mathcal{A}}(\sigma)$ is an isomorphism for any open subset U of X and any $\sigma \in \mathcal{A}(U)$.

Example A.1.4. If A is a k-algebra, we denote by A^+ the k-linear category with one object and having A as the endomorphism ring of this object. Let $\mathcal{A}$ be a sheaf of algebras on X and consider the prestack which associates to an open set U the category $\mathcal{A}(U)^+$. The associated stack to this prestack, denoted $\mathcal{A}^\dagger$, is an algebroid stack.

For two k-linear category $\mathscr{C}$ and $\mathscr{C}'$ one defines their tensor product $\mathscr{C} \otimes \mathscr{C}'$ by setting $\mathrm{Ob}(\mathscr{C} \otimes \mathscr{C}') = \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C}')$ and

$$\mathrm{Hom}_{\mathscr{C} \otimes \mathscr{C}'}((M, M'), (N, N')) = \mathrm{Hom}_{\mathscr{C}}(M, N) \otimes \mathrm{Hom}_{\mathscr{C}'}(M', N').$$

Remark A.1.5. The tensor product of two algebroid stacks is well-defined. If $\mathcal{A}$ and $\mathcal{A}'$ are two algebroid stacks their tensor product is the k-linear stack associated with the prestack $U \mapsto \mathcal{A}(U) \otimes \mathcal{A}'(U)$ where U is an open subset of X.

An algebroid stacks can be described by what is called gluing datum. To define them, we need to introduce the notion of bi-invertible-module.

Definition A.1.6. Let $\mathcal{R}$ and $\mathcal{R}'$ be two sheaves of k-algebras. An $\mathcal{R} \otimes \mathcal{R}'$-module $\mathcal{L}$ is called bi-invertible if there exists locally a section w of $\mathcal{L}$ such that the morphisms $\mathcal{R} \ni r \mapsto (r \otimes 1)w \in \mathcal{L}$ and $\mathcal{R}' \ni r' \mapsto (1 \otimes r')w \in \mathcal{L}$ give isomorphisms of $\mathcal{R}$-modules and $\mathcal{R}'$-modules, respectively.

Let $\{U_i\}_{i \in I}$ be an open covering of X, we set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, etc.

Definition A.1.7. An algebroid datum on $\mathcal{U} = \{U_i\}_{i \in I}$ is the data of

- a k-algebroid $\mathcal{A}$ on X,
- for every $U_i \in \mathcal{U}$, $\sigma_i \in \mathcal{A}(U_i)$,
- for every U_{ij}, an isomorphism $\phi_{ij} : \sigma_j|_{U_{ij}} \xrightarrow{\sim} \sigma_i|_{U_{ij}}$.

Definition A.1.8. A gluing datum on $\mathcal{U} = \{U_i\}_{i \in I}$ is the data of

- for every $U_i \in \mathcal{U}$ a sheaf $\mathcal{A}_i$ of k-algebras on U_i
- for every U_{ij} a bi-invertible $\mathcal{A}_i \otimes \mathcal{A}_j^{op}$-module $\mathcal{L}_{ij}$ on U_{ij}.
- for every U_{ijk} an isomorphism $a_{ijk} : \mathcal{L}_{ij} \otimes_{\mathcal{A}_j} \mathcal{L}_{jk} \xrightarrow{\sim} \mathcal{L}_{ik}$ in $\mathrm{Mod}(\mathcal{A}_i \otimes \mathcal{A}_k^{op}|_{U_{ijk}})$.

such that the diagram below in $\mathrm{Mod}(\mathcal{A}_i \otimes \mathcal{A}_k^{op}|_{U_{ijkl}})$ commutes:

$$\begin{tikzcd}
\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{kl} \arrow[r, "a_{ijk}"] \arrow[d, "a_{jkl}"] & \mathcal{L}_{ik} \otimes \mathcal{L}_{kl} \arrow[d, "a_{ikl}"] \\
\mathcal{L}_{ij} \otimes \mathcal{L}_{jl} \arrow[r, "a_{ijl}"] & \mathcal{L}_{il}
\end{tikzcd}$$

Proposition A.1.9. *Consider a gluing datum on $\mathcal{U}$. Then there exist an algebroid datum to which this gluing datum is associated. This algebroid datum is unique up to an equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

A.2 Module over an algebroid stack

We denote by k_X the constant sheaf with coefficient in k, by $\mathfrak{Mod}(k_X)$ the k-linear stack of sheaves of k-modules on X. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two k-linear stacks, we denote by $\mathrm{Fct}_k(\mathfrak{S}_1, \mathfrak{S}_2)$ the category of k-linear functors of stacks from $\mathfrak{S}_1$ to $\mathfrak{S}_2$. We set

$$\mathrm{Mod}(\mathcal{A}) = \mathrm{Fct}_k(\mathcal{A}, \mathfrak{Mod}(k_X))$$

Proposition A.2.1. *For a k-algebroid $\mathcal{A}$, the k-linear prestack $U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$ is a stack and we denote it $\mathfrak{Mod}(\mathcal{A})$.*

If $\mathfrak{S}$ is a stack on X we denote by $\mathfrak{S}^{op}$ the stack defined by $\mathfrak{S}^{op}(U) := \mathfrak{S}(U)^{op}$.

Remark A.2.2. An algebroid stack $\mathcal{A}$ is not in general an object of $\mathrm{Mod}(\mathcal{A})$. But, it can be canonically identified to an object of $\mathrm{Mod}(\mathcal{A} \otimes \mathcal{A}^{op})$ via its morphisms sheaf $\mathcal{H}om_{\mathcal{A}}(\cdot, \cdot)$ as follows

$$\mathcal{A} \otimes \mathcal{A}^{op} \ni (\sigma, \tau) \mapsto \mathcal{H}om_{\mathcal{A}}(\tau, \sigma) \in \mathfrak{Mod}(k_X).$$

Definition A.2.3. An $\mathcal{A}$-module is invertible if for any open subset U of X and any $\sigma \in \mathcal{A}(U)$, the $\mathcal{E}nd(\sigma)$-module $\mathcal{L}(\sigma)$ is locally isomorphic to $\mathcal{E}nd(\sigma)$. We denote by $\mathrm{Inv}(\mathcal{A})$ the full subcategory of $\mathrm{Mod}(\mathcal{A})$ consisting of invertible $\mathcal{A}$-module.

Proposition A.2.4. *If $\mathfrak{Inv}(\mathcal{A})$ denote the full substack of invertible modules of $\mathfrak{Mod}(\mathcal{A})$ then we have an equivalence of k-linear stacks*

$$\mathcal{A} \xrightarrow{\sim} \mathfrak{Inv}(\mathcal{A}^{op}) \xrightarrow{\sim} \mathfrak{Inv}(\mathcal{A})^{op}. \tag{A.2.1}$$

We give a description of modules over an algebroid stack in term of local data.

Let $\mathcal{A}$ be an algebroid stack on X. We can associate to $\mathcal{A}$ a gluing datum as follows. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X such that for every $i \in I$ there is a $\sigma_i \in \mathcal{A}(U_i)$ and choose one of these σ_i for every $i \in I$.

To these data, we associate

- $\mathcal{A}_i := \mathcal{E}nd(\sigma_i)$,
- $\mathcal{L}_{ij} := \mathcal{H}om_{\mathcal{A}_i|_{U_{ij}}}(\sigma_j|_{U_{ij}}, \sigma_i|_{U_{ij}})$, (Then $\mathcal{L}_{ij}$ is a bi-invertible $\mathcal{A}_i \otimes \mathcal{A}_j^{op}$ module on U_{ij}),
- the natural isomorphism

$$a_{ijk} : \mathcal{L}_{ij} \otimes_{\mathcal{A}_j} \mathcal{L}_{ij} \xrightarrow{\sim} \mathcal{L}_{ik} \text{ in } \mathrm{Mod}(\mathcal{A}_i \otimes \mathcal{A}_k^{op}|_{U_{ijk}}).$$

Proposition A.2.5. *To specify a module over $\mathcal{A}$, it is sufficient to give a family $\{\mathcal{M}_i, q_{ij}\}_{i,j \in I}$ with $\mathcal{M}_i \in \mathrm{Mod}(\mathcal{A}_i)$ and the q_{ij} are isomorphisms*

$$q_{ij} : \mathcal{L}_{ij} \otimes_{\mathcal{A}_{ij}} \mathcal{M}_j \xrightarrow{\sim} \mathcal{M}_i$$

making the diagram commutative

$$\begin{tikzcd}
\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{M}_k \arrow[r, "q_{jk}"] \arrow[d, "a_{ijk}"] & \mathcal{L}_{ij} \otimes \mathcal{M}_j \arrow[d, "q_{ij}"] \\
\mathcal{L}_{ik} \otimes \mathcal{M}_k \arrow[r, "q_{ik}"] & \mathcal{M}_i.
\end{tikzcd}$$

A similar description can be given for morphisms of $\mathcal{A}$-modules.

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List of Notations